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SHAKEDOWN AND STEADY-STATE RESPONSES OF ELASTIC–PLASTIC SOLIDS IN LARGE DISPLACEMENTS

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Abstract—Elastic-perfectly plastic solids (or structures) subjected to loads quasi-statically varying within a specified domain are addressed in the framework of large displacements and the additive strain decomposition rule. On the ground of Drucker's principle of stability in the large, an appropriate stability requisite (called D-stability) is formulated as the positive definiteness property of a specific functional, sum of the second variation of the Helmholtz free energy with an additional term depending on higher-order geometry change effects. For a D-stable structure for which the additive strain decomposition rule is applicable, Melan's and Koiter's theorems of classical shakedown theory are reconsidered and reformulated for large displacements. The extended Melan and Koiter theorems so established save the essential features of the classical ones, but exhibit a greater formal complexity with consequent difficulties for engineering applications. For structures subjected to periodic loads, it is shown that—as long as the structure does not incur loss of D-stability—a long-term steady-state response (or steady cycle) occurs, which exhibits the same periodicity characteristics as in case of small displacements; that is, the (second Piola–Kirchhoff) stresses and the plastic (Green–Lagrange) strain rates become periodic as the load. A few illustrative numerical results are presented. Copyright © 1996 Elsevier Science Ltd

NOTATION

A compact notation is used throughout, with vectors and tensors denoted by bold face symbols. The notation $\mathbf{A}\cdot\mathbf{B}$ means the two-index contraction operation between two tensors **A** and **B**, for instance if **A** and **B** are second-order tensors with components A_{ik} and B_{kj} it is $(\mathbf{A}\cdot\mathbf{B})_{ij} = A_{ik}B_{kj}$, where the repeated index summation rule holds. Analogously the notation $\mathbf{A}:\mathbf{B}$ means the four index contraction operation between the tensors **A** and **B**, for instance if **A** and **B** are third-order tensors it is $(\mathbf{A}:\mathbf{B})_{ij} = A_{ikh}B_{hkj}$. Also, $\mathbf{A}^T$ = transpose of **A**, **I** = unit tensor. The Lagrangian description is used with material Cartesian orthogonal coordinates $\mathbf{X} = \{X_I, I = 1, 2, 3\}$ and spatial coordinates $\mathbf{x} = \{x_i, i = 1, 2, 3\}$. A superposed dot means time derivative. The symbol $:=$ denotes equality by definition. The following two differential operators are widely used, namely:

$\nabla(\cdot) :=$ gradient of $(\cdot)$, a tensor whose components are the partial derivatives of $(\cdot)$; e.g., if **u** has components u_i, then $\nabla\mathbf{u}$ has components $\partial u_i/\partial X_J = u_{i,J}$;

$\mathrm{div}(\cdot) :=$ divergence of $(\cdot)$; e.g. if **S** is a second-order tensor with components S_{IJ} then div **S** has components $\partial S_{JI}/\partial X_J = S_{JI,J}$.

Other notations and rules will be defined at their first appearance in the text.

1. INTRODUCTION

Shakedown theory in large, or moderately large, displacements has received attention in recent years. After the pioneering works of Davies (1967) and Gavarini and Beolchini (1970)—who independently studied the influence of geometry changes on the shakedown limit load for some simple frame structures—and after the work of Maier (1973)—who introduced a new class of shakedown problems for pre-stressed structures and extended Melan's and Koiter's theorems as to include so-called second-order geometric effects—several attempts were made for a more inclusive generalization of the above shakedown theorems to geometric nonlinearities. Weichert (1984, 1986) addressed this problem successfully with the use of the additive strain decomposition rule, but his results hold under severe limitations on the actual structural response. Gross-Weege (1990), using the same strain decomposition rule, provided an extended Melan's theorem for structures subjected to a constant load, responsible for large displacements, and to additional variable loads causing "small" additional displacements. The multiplicative strain decomposition rule was used by Tritsch and Weichert (1992), who provided a sufficient Melan-type statement for shakedown on the basis of some simplifying hypotheses on the additivity of some elastic and residual elastic strain rates. Stumpf (1993, 1994), employed the multiplicative strain decomposition rule and attempted to reformulate Melan's theorem stating that shakedown occurs if there exists some "real self-equilibrated residual state" of the structure under a given load history, but no criterion, other than a numerical one, is provided for *a priori* recognizing whether such a state exists or not. Maier *et al.* (1993) studied, through numerical analysis procedures, the effects of geometry changes and of buckling on shakedown and ratchetting of a circular cylindrical shell subjected to thermal load cycles and a constant axial force and pointed out how these effects may be dangerous for design purposes, but no theoretical results of general validity are therein presented.

In consideration of the above, it seems legitimate to state that a consistent large displacement shakedown theory, as satisfactory as the classical one, is still lacking. Primarily, in the authors' opinion, there has apparently been no adequate criticism of the Melan theorem's central concept (i.e. the concept of time-independent self-stresses to be superposed to elastic stresses such as to generate plastically admissible stresses) in relation to its validity in the large displacements framework. In effect, in a structure in which (large-displacement) shakedown has already occurred, the stress state in any subsequent time can always be expressed as the sum of the elastic stresses (i.e. computed on the basis of elasticity theory for large displacements) due to the loads, with the self-equilibrated stresses caused, upon the loaded body considered elastic, by the plastic strains produced before shakedown, but the latter stresses cannot be time independent since they must comply with equilibrium in a body's configuration which changes with time. As a consequence, time-independent self-stresses of classical Melan theorem must be replaced by time-independent plastic strains to be superposed as initial strains to the assigned loads.

Analogous consideration can be developed in relation to the Koiter theorem's central concept (i.e. the concept of kinematically admissible plastic strain cycle resulting in a compatible strain field). Namely, such a plastic strain cycle can be viewed as an imposed plastic strain history applied upon the structure considered elastic and subjected to a given potentially active load history. For large displacements, the above plastic strain cycle cannot be applied on the unloaded body, as it is usual for small displacements, but rather it must act in conjunction with the load history, and the resulting applied plastic strains must satisfy the compatibility requirement in the body's deformed configuration at the end of the load cycle.

A series of papers [see e.g. König (1982), Nguyen (1984), Siemaszko and König (1985)] studied global destabilization that may be induced by progressive plastic deformation in case of ratchetting. But, apart from this and from the numerical analyses of Maier *et al.* (1993), it seems that crucial aspects of the structural behavior, such as equilibrium stability and buckling, have had a minor influence, if not at all, on the proposed shakedown formulations.

The main purpose of the present paper is to address shakedown for large (or finite) displacements with the use of the additive strain decomposition rule. The multiplicative strain decomposition rule would be more appropriate to this aim, however, kinematics of finite plastic deformation and the related constitutive equations still have many controversial aspects [see, e.g. Lee (1981), Nemat-Nasser (1981, 1992), Simo and Ortiz (1985), Simo (1988), Naghdi (1990), Foerster and Kuhn (1994), Stumpf (1994)] and deserve definitive clarification before being incorporated into a firm shakedown theory. On the other hand, the use of the additive strain decomposition rule has a two-fold justification. First, such a rule is valid within a wide range of strain and displacement approximations of practical importance characterized by small strains and moderate rotations (pin-jointed structures, beam and frame structures, thin plates and shells, but here no approximation of any sort is explicitly introduced) [see Casey (1985), Weichert (1986) and De Tommasi and Marzano (1993)]. Secondly, some essential features of a large-displacement shakedown theory are independent of the strain decomposition rule therein employed.

In this way, Melan's and Koiter's theorems will be reconsidered and restated in the framework of large displacements. (The term "large" is here preferred to the widely used term "finite" to signify that the theory of finite deformation has been only partially applied here.) Additionally, the asymptotic response of the body subjected to periodically variable loads will be studied in order to show the conditions under which there may exist a stabilized long-term response (or steady cycle) characterized by periodic (second Piola–Kirchhoff) stresses and plastic (Green–Lagrange) strain rates with the same period as the loads.

The plan of the paper is the following. After some preliminaries (Section 2), an appropriate stability requisite is established in Section 3, and the extended Melan's and Koiter's theorems are presented in Sections 4 and 5, respectively. Section 6 is devoted to the assessment of the condition for the existence of the steady cycle in case of periodic loads. Section 7 examines the case of lack of standard stability in a D-stable structure. Section 8 presents a few numerical illustrative examples, and finally Section 9 reports the conclusions.

2. DEFINITIONS AND PRELIMINARIES

The initial configuration of a continuous solid body B is described by Cartesian orthogonal coordinates $\mathbf{X} = (X_1, X_2, X_3)$. V is the (open) region initially occupied by the body, with boundary surface $\partial V = \partial_D V \cup \partial_T V$, $\partial_D V \cap \partial_T V = \emptyset$, where $\partial_D V$ denotes the portion of ∂V where displacements are prescribed. The body undergoes a time-continuous configuration change, with the current configuration referred to the same Cartesian axes as the initial one, and described by the spatial co-ordinate $\mathbf{x} = \mathbf{X} + \mathbf{u}$. $\mathbf{u} = \mathbf{u}(\mathbf{X}, t)$ is the displacement vector of the particle $\mathbf{X}$ at time $t \geqslant 0$, having (sufficiently regular) components $u_i(\mathbf{X}, t) = u_I(\mathbf{X}, t)$, $(i = I = 1, 2, 3)$, such that $\mathbf{u}(\mathbf{X}, 0) = \mathbf{0}$ for all $\mathbf{X}$.

The transformation $\mathbf{X} \to \mathbf{x}$ is described by the deformation gradient $\mathbf{F}$, i.e.

$$\mathbf{F} := \partial \mathbf{x}/\partial \mathbf{X} = \mathbf{I} + \nabla \mathbf{u}. \tag{1}$$

$\mathbf{F}$ has components $F_{iJ} = \partial x_i/\partial X_J = \delta_{iJ} + u_{i,J}$ where $u_{i,J} := \partial u_i/\partial X_J$. The body's strain state in the spatial configuration $\mathbf{x}$ is measured by the Green–Lagrange strain tensor $\mathbf{E} = \{E_{IJ}\}$, i.e.

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}) \tag{2a}$$

$$= \tfrac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T + (\nabla \mathbf{u})^T \cdot \nabla \mathbf{u}]. \tag{2b}$$

For subsequent use, $\mathbf{E}$ is decomposed as $\mathbf{E} = \mathbf{E}^{(1)} + \mathbf{E}^{(2)}$, where $\mathbf{E}^{(1)}$ is its linear part, $\mathbf{E}^{(2)}$ is its nonlinear one, i.e.

$$\mathbf{E}^{(1)} := \tfrac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}}] \tag{2c}$$

$$\mathbf{E}^{(2)} := \tfrac{1}{2}(\nabla \mathbf{u})^{\mathrm{T}} \cdot \nabla \mathbf{u}. \tag{2d}$$

By hypothesis, the *additive* decomposition rule holds for E, that is

$$\mathbf{E} = \mathbf{E}^{\mathrm{e}} + \mathbf{E}^{\mathrm{p}} \tag{3}$$

where $\mathbf{E}^{\mathrm{e}}$ and $\mathbf{E}^{\mathrm{p}}$ stand for elastic and plastic parts of $\mathbf{E}$, respectively. $\mathbf{E}^{\mathrm{e}}$ is related to the (symmetric) second Piola–Kirchhoff stress tensor $\mathbf{S} = \{S_{IJ}\}$ by Hooke's law, i.e.

$$\mathbf{S} = \mathbf{C} : \mathbf{E}^{\mathrm{e}} \tag{4}$$

where $\mathbf{C} = \{C_{IJHK}\}$ is the (isothermal) elastic moduli tensor (with its usual symmetries). $\dot{\mathbf{E}}^{\mathrm{p}}$ complies with the flow laws of associated plasticity for rate-independent perfectly plastic materials, which are here assumed as follows:

$$\dot{\mathbf{E}}^p = \dot{\lambda}\frac{\partial \phi}{\partial \mathbf{S}} \tag{5}$$

$$\phi(\mathbf{S}) \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \dot{\lambda}\phi(\mathbf{S}) = 0 \tag{6}$$

where $\phi = \phi(\mathbf{S})$ is the relevant (convex and smooth) yield function and $\dot{\lambda}$ is the relevant plastic coefficient. $D(\dot{\mathbf{E}}^p)$ is the related dissipation function, such that $\mathbf{S} = \partial D/\partial \dot{\mathbf{E}}^p$ provides the stress corresponding to a nonvanishing $\dot{\mathbf{E}}^p$ through the flow rules. The body is subjected to volume forces $\mathbf{b} = \{b_i(\mathbf{X}, \mathbf{P})\}$ (per unit undeformed volume in V) and to surface forces $\mathbf{f} = \{f_i(\mathbf{X}, \mathbf{P})\}$ (per unit undeformed surface over $\partial_T V$), where $\mathbf{P}$ is the vector of the independent load parameters. For simplicity, kinematical external actions as imposed strains in V and imposed displacements on $\partial_D V$ are assumed to be vanishing; also, temperature is taken constant. $\mathbf{P}$ is allowed to range within some (finite) domain Π, called *load domain*, belonging to an Euclidian space of adequate dimensions (but, without loss of generality, it is assumed two-dimensional in the following). Any load path $\mathbf{P}(t) \in \Pi$, $0 \leqslant t \leqslant T$, is a potentially active load history (*admissible load history*, ALH).

At any instant $t \in (0, T)$, $\mathbf{E}$ must be compatible with $\mathbf{u}$, that is eqn (2a), or (2b), must be satisfied everywhere in V with the boundary condition $\mathbf{u} = \mathbf{0}$ on $\partial_D V$, whereas $\mathbf{S}$ must satisfy equilibrium with the applied loads upon the current configuration $\mathbf{x} = \mathbf{X} + \mathbf{u}(\mathbf{X}, t)$, i.e.

$$\operatorname{div}(\mathbf{S} \cdot \mathbf{F}^T) + \mathbf{b} = \mathbf{0} \quad \text{in } V, \tag{7a}$$

$$\mathbf{n} \cdot \mathbf{S} \cdot \mathbf{F}^T = \mathbf{f} \quad \text{on } \partial_T V, \tag{7b}$$

where $\mathbf{n}$ is the unit external normal to ∂V. These equilibrium conditions can also be expressed by the virtual work principle. Namely, let $\delta\mathbf{u} = \delta\mathbf{u}(\mathbf{X})$ denote virtual displacements of the material points $\mathbf{X}$ from their respective spatial position $\mathbf{x} = \mathbf{X} + \mathbf{u}$ with $\delta\mathbf{u} = \mathbf{0}$ on $\partial_D V$, and let

$$\delta\mathbf{F} := \nabla(\delta\mathbf{u}), \quad \delta\mathbf{E} := \tfrac{1}{2}(\mathbf{F}^T \cdot \delta\mathbf{F} + \delta\mathbf{F}^T \cdot \mathbf{F}) \tag{8}$$

be the related virtual deformation gradient and Green–Lagrange strain fields. Then the identity (9) holds for arbitrary choices of the virtual displacements $\delta\mathbf{u}$, but with $\delta\mathbf{u} = \mathbf{0}$ on $\partial_D V$ (Washizu, (1982), namely

$$\int_V \mathbf{b} \cdot \delta\mathbf{u}\, \mathrm{d}V + \int_{\partial_T V} \mathbf{f} \cdot \delta\mathbf{u}\, \mathrm{d}S = \int_V \mathbf{S} : \delta\mathbf{E}\, \mathrm{d}V \tag{9}$$

where $\mathrm{d}V$ and $\mathrm{d}S$ denote volume and surface elements, respectively.

Assuming e.g. that $\mathbf{E}^p \equiv \mathbf{0}$ at $t = 0$, the elastic–plastic response of the body to a specified ALH can in principle be obtained through eqns (2)–(7) (possibly with the aid of an adequate displacement control). The related body's motion $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, $\mathbf{X} \in V$, $0 \leqslant t \leqslant T$, is referred to as an equilibrium path in the following.

3. D-STABILITY

For an elastic structure subjected to conservative loads, a stable (noncritical) state is, according to the Lagrange–Dirichlet theorem (Thomson and Hunt, 1973; Bažant and Cedolin, 1991), an equilibrium state at which the second variation of the total potential energy is positive, and a stable equilibrium path is a continuous sequence of such stable states. For an elastic–plastic structure the above energy criterion of stability can be applied with reference to the *tangentially equivalent* elastic structure and using the related free energy (Bažant and Cedolin, 1991).

In view of the scope of the present paper, the Drucker principle of stability in the large (Drucker, 1960) is employed in order to establish a stability condition particularly suitable to the subsequent developments. To this aim, let $\mathbf{x}^{(a)} = \mathbf{x}^{(a)}(\mathbf{X}, t)$ be a (fundamental) equilibrium path related to some load history $\mathbf{P}^{(a)}(t)$, $t \geqslant 0$, with volume and surface forces $\mathbf{b}^{(a)}(\mathbf{X}, t)$ and $\mathbf{f}^{(a)}(\mathbf{X}, t)$, and let $\mathbf{u}^{(a)}(\mathbf{X}, t)$, $\mathbf{F}^{(a)}(\mathbf{X}, t)$, etc. denote the related response variables. Let an external agency slowly apply a nonsmall load perturbance $\Delta\mathbf{P}(t)$, $t \geqslant 0$, and possibly an initial plastic strain field $\Delta\bar{\mathbf{E}}^{\mathrm{p}}(\mathbf{X})$. The augmented load history $\mathbf{P}^{(b)}(t) = \mathbf{P}^{(a)}(t) + \Delta\mathbf{P}(t)$ produces a (perturbed) equilibrium path $\mathbf{x}^{(b)} = \mathbf{x}^{(b)}(\mathbf{X}, t)$ with response variables $\mathbf{u}^{(b)}(\mathbf{X}, t)$, $\mathbf{F}^{(b)}(\mathbf{X}, t)$, etc. The Drucker principle for stability in the large can be expressed stating that the work, ΔL_e, performed by the perturbance loads $\Delta\mathbf{P}$ through the difference response within any time interval $(0, t_1)$ is positive, i.e.

$$\Delta L_e := \int_0^{t_1} \left\{ \int_V \Delta\mathbf{b} \cdot (\dot{\mathbf{u}}^{(b)} - \dot{\mathbf{u}}^{(a)})\,\mathrm{d}V + \int_{\partial_T V} \Delta\mathbf{f} \cdot (\dot{\mathbf{u}}^{(b)} - \dot{\mathbf{u}}^{(a)})\,\mathrm{d}S \right\} \mathrm{d}t > 0, \tag{10}$$

which is to be satisfied for any choice of the perturbance within some perturbance limits.

Before drawing any consequence from eqn (10), let the compatibility and equilibrium conditions be explicitly written in terms of increments of the response variables, that is, of quantities as $\Delta(\cdot) := (\cdot)^{(b)} - (\cdot)^{(a)}$, e.g. $\Delta\mathbf{u} = \mathbf{u}^{(b)} - \mathbf{u}^{(a)}$, $\Delta\mathbf{F} = \mathbf{F}^{(b)} - \mathbf{F}^{(a)} = \nabla\mathbf{u}^{(b)} - \nabla\mathbf{u}^{(a)}$, $\Delta\mathbf{E} = \mathbf{E}^{(b)} - \mathbf{E}^{(a)}$, etc. So, the following eqns (11) and (12) can be easily proved to hold, i.e.

$$\Delta\mathbf{E} = \Delta^{(1)}\mathbf{E} + \Delta^{(2)}\mathbf{E} \tag{11a}$$

where

$$\Delta^{(1)}\mathbf{E} := \tfrac{1}{2}(\mathbf{F}^{(a)\mathrm{T}} \cdot \Delta\mathbf{F} + \Delta\mathbf{F}^{\mathrm{T}} \cdot \mathbf{F}^{(a)}) \tag{11b}$$

$$\Delta^{(2)}\mathbf{E} := \tfrac{1}{2}\Delta\mathbf{F}^{\mathrm{T}} \cdot \Delta\mathbf{F} \tag{11c}$$

and

$$\Delta\dot{\mathbf{E}} = \Delta^{(1)}\dot{\mathbf{E}} + \Delta^{(2)}\dot{\mathbf{E}} \tag{12a}$$

where

$$\Delta^{(1)}\dot{\mathbf{E}} := \tfrac{1}{2}(\mathbf{F}^{(a)\mathrm{T}} \cdot \Delta\dot{\mathbf{F}} + \Delta\dot{\mathbf{F}}^{\mathrm{T}} \cdot \mathbf{F}^{(a)}) \tag{12b}$$

$$\Delta^{(2)}\dot{\mathbf{E}} = \tfrac{1}{2}(\Delta\mathbf{F}^{\mathrm{T}} \cdot \Delta\dot{\mathbf{F}} + \Delta\dot{\mathbf{F}}^{\mathrm{T}} \cdot \Delta\mathbf{F} + \dot{\mathbf{F}}^{(a)\mathrm{T}} \cdot \Delta\mathbf{F} + \Delta\mathbf{F}^{\mathrm{T}} \cdot \dot{\mathbf{F}}^{(a)}). \tag{12c}$$

Analogously, the stress increment $\Delta\mathbf{S} = \mathbf{S}^{(b)} - \mathbf{S}^{(a)}$ is easily shown to satisfy the equilibrium equations:

$$\operatorname{div}(\Delta\mathbf{S} \cdot \mathbf{F}^{(a)\mathrm{T}}) + \Delta\mathbf{b}^* = \mathbf{0} \quad \text{in } V \tag{13a}$$

$$\mathbf{n} \cdot \Delta\mathbf{S} \cdot \mathbf{F}^{(a)\mathrm{T}} = \Delta\mathbf{f}^* \quad \text{on } \partial_T V \tag{13b}$$

where

$$\Delta\mathbf{b}^* := \Delta\mathbf{b} + \operatorname{div}(\mathbf{S}^{(a)} \cdot \Delta\mathbf{F}^{\mathrm{T}}) + \operatorname{div}(\Delta\mathbf{S} \cdot \Delta\mathbf{F}^{\mathrm{T}}) \tag{14a}$$

$$\Delta\mathbf{f}^* := \Delta\mathbf{f} - \mathbf{n} \cdot \mathbf{S}^{(a)} \cdot \Delta\mathbf{F}^{\mathrm{T}} - \mathbf{n} \cdot \Delta\mathbf{S} \cdot \Delta\mathbf{F}^{\mathrm{T}}. \tag{14b}$$

In other words, the stress increment field $\Delta\mathbf{S}(\mathbf{X}, t)$ is in equilibrium, upon the configuration $\mathbf{x}^{(a)} = \mathbf{X} + \mathbf{u}^{(a)}$, with the volume forces $\Delta\mathbf{b}^*$ in V and surface forces $\Delta\mathbf{f}^*$ on $\partial_T V$, superpositions of the assigned load perturbances, $\Delta\mathbf{b}$ and $\Delta\mathbf{f}$, with the consequent higher-order geometric-effect forces, namely, the second and third terms of the right-hand side of eqns (14a, b).

Since $\Delta\dot{\mathbf{u}} = \mathbf{0}$ on $\partial_D V$, $\Delta\dot{\mathbf{u}}$ can be viewed as a virtual displacement field applied upon the configuration $\mathbf{x}^{(a)} = \mathbf{X} + \mathbf{u}^{(a)}$, with the related virtual deformation gradient $\Delta\dot{\mathbf{F}} = \nabla(\Delta\dot{\mathbf{u}})$

and virtual strains $\Delta^{(1)}\dot{\mathbf{E}}$ given by eqn (12b) [see eqn (8)]. Then, by the virtual work principle (9) and the equilibrium eqns (13) and (14), the following can be shown to hold, i.e.

$$\Delta L_e = \int_0^{t_1}\left[\int_V \Delta\mathbf{S}:\Delta^{(1)}\dot{\mathbf{E}}\,\mathrm{d}V - \int_V \operatorname{div}[(\mathbf{S}^{(a)}+\Delta\mathbf{S})\cdot\Delta\mathbf{F}^{\mathrm{T}}]\cdot\Delta\dot{\mathbf{u}}\,\mathrm{d}V + \int_{\partial_T V}\mathbf{n}\cdot(\mathbf{S}^{(a)}+\Delta\mathbf{S})\cdot\Delta\mathbf{F}^{\mathrm{T}}\cdot\Delta\dot{\mathbf{u}}\,\mathrm{d}S\right]\mathrm{d}t. \quad (15)$$

This equation, by the divergence theorem and taking into account eqns (12a, c) can also be written as

$$\Delta L_e = \int_0^{t_1}\left[\int_V \Delta\mathbf{S}:\Delta\dot{\mathbf{E}}\,\mathrm{d}V - \int_V \Delta\mathbf{S}:(\Delta\mathbf{F}^{\mathrm{T}}\cdot\Delta\dot{\mathbf{F}}+\Delta\mathbf{F}^{\mathrm{T}}\cdot\dot{\mathbf{F}}^{(a)})\,\mathrm{d}V + \int_V [(\mathbf{S}^{(a)}+\Delta\mathbf{S})\cdot\Delta\mathbf{F}^{\mathrm{T}}]:\Delta\dot{\mathbf{F}}\,\mathrm{d}V\right]\mathrm{d}t. \quad (16)$$

Then, substituting from eqn (3), using eqn (4) and with the positions

$$W_{\mathrm{F}}(t_1) := \frac{1}{2}\int_V \Delta\mathbf{S}:\mathbf{C}^{-1}:\Delta\mathbf{S}\,\mathrm{d}V\bigg|_{t_1} + \int_V \mathbf{S}^{(a)}:\Delta^{(2)}\mathbf{E}\,\mathrm{d}V\bigg|_{t_1} \quad (17)$$

$$W_{\mathrm{G}}(t_1) := -\int_0^{t_1}\left[\int_V (\Delta\mathbf{S}\cdot\Delta\mathbf{F}^{\mathrm{T}}):\dot{\mathbf{F}}^{(a)}\,\mathrm{d}V + \int_V \Delta^{(2)}\mathbf{E}:\dot{\mathbf{S}}^{(a)}\,\mathrm{d}V\right]\mathrm{d}t, \quad (18)$$

$$W(t) := W_{\mathrm{F}}(\mathbf{t}) + W_{\mathrm{G}}(t), \quad (19)$$

and assuming $W(0) = 0$ (this condition is achieved if $\Delta\mathbf{P}(0) = \mathbf{0}$ and $\Delta\bar{\mathbf{E}}^{\mathrm{p}} \equiv \mathbf{0}$), inequality (10) can be given the expression

$$\Delta L_e = W(t_1) + \int_0^{t_1}\int_V \Delta\mathbf{S}:\Delta\dot{\mathbf{E}}^{\mathrm{p}}\,\mathrm{d}V\,\mathrm{d}t > 0. \quad (20)$$

Note that $W_F(t)$ is the second "finite" variation of the functional

$$\Psi = \frac{1}{2}\int_V \mathbf{E}^{\mathrm{e}(a)}:\mathbf{C}:\mathbf{E}^{\mathrm{e}(a)}\,\mathrm{d}V - \int_V \mathbf{b}\cdot\mathbf{u}^{(a)}\,\mathrm{d}V - \int_{\partial_T V}\mathbf{f}\cdot\mathbf{u}^{(a)}\,\mathrm{d}S \quad (21)$$

evaluated at t and representing the Helmholtz free energy of the "tangentially equivalent" elastic system, that is, of the given elastic–plastic body with the accumulated plastic strains $\mathbf{E}^{\mathrm{p}(a)}$. Since small perturbances can also be considered, $W_{\mathrm{F}} > 0$ implies—by the Lagrange–Dirichlet theorem—standard stability (i.e. stability in the usual sense) of the given structure. Moreover, W_{G} is a functional depending on certain higher-order geometric effects produced by the perturbance.

Inequality (20) holds good also if both equilibrium paths $\mathbf{x}^{(a)}$ and $\mathbf{x}^{(b)}$ are elastic, in which case $\dot{\mathbf{E}}^{\mathrm{p}(a)} = \dot{\mathbf{E}}^{\mathrm{p}(b)} = \Delta\dot{\mathbf{E}}^{\mathrm{p}} = \mathbf{0}$ identically and eqn (20) reduces to $W(t) > 0$, $\forall t > 0$. If, on the other hand, one or both equilibrium paths exceed the elastic range and eqn (20) is satisfied, the last term of eqn (20) being always nonnegative by the material stability postulate (Martin, 1975; Lubliner, 1990), the functional $W(t)$ may take negative values; on the contrary, if $W(t) > 0\,\forall t > 0$, then eqn (20) is certainly satisfied. This enables one to introduce a sufficient criterion for stability in the large in the Drucker sense, or D-stability for brevity.

The (fundamental) equilibrium path $\mathbf{x}^{(a)} = \mathbf{x}^{(a)}(\mathbf{X}, t)$ of the structure is, by definition, qualified D-stable if the D-stability functional $W(t)$, as defined by eqns (17)–(19), is positive definite in some set of neighbor equilibrium paths $\mathbf{x}^{(b)} = \mathbf{x}^{(b)}(\mathbf{X}, t)$, that is for all $\mathbf{x}^{(b)}(\mathbf{X}, t)$ satisfying an inequality as $\|\mathbf{x}^{(b)}(\mathbf{X}, t) - \mathbf{x}^{(a)}(\mathbf{X}, t)\| \leqslant \eta$ where $\|\cdot\|$ is a suitable norm and η some scalar, every $\mathbf{x}^{(b)}(\mathbf{X}, t)$ being generated with the aid of a suitable perturbance $\Delta\mathbf{P}(t)$ (possibly with initial plastic strains $\Delta\bar{\mathbf{E}}^{p}$). Obviously, any piece of a D-stable equilibrium path is also D-stable. The equilibrium path $\mathbf{x}^{(a)}(\mathbf{X}, t)$ is only "partially" D-stable if its queue $t \geqslant t_0$ for some t_0 is not D-stable.

Analogously, a structure is qualified as D-stable if its equilibrium paths $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, promoted by any load history and initial conditions within some limits, are each D-stable. If some one of these equilibrium paths is partially D-stable, the structure itself is referred to as partially D-stable; this may be the case e.g. for a structure suffering a straining process with progressive plastic strain accumulation (ratchetting) under cyclic loads.

The above definition of the D-stability for a structure provides a condition that may be more restrictive than standard stability ($W_F > 0$) due to the negative contribution of W_G in eqn (19) for certain neighbor equilibrium paths. Here, the assumption is made that the D-stability implies standard stability; that is, $W(t) > 0$ implies $W_F(t) > 0$. (We will resort to this point later on in this paper, see Section 7.) If all the neighbor equilibrium paths are infinitesimally close to the fundamental equilibrium path (i.e. they can be thought of as generated by "small" perturbances), such that $\Delta^{(2)}\mathbf{E}$ and $\Delta\mathbf{S}\cdot\mathbf{F}^{T}$ in eqn (18) can be approximated to zero, then $W_G(t)$ drops from eqn (19) and $W(t) = W_F(t)$ = second variation of Helmholtz free energy at time t; that is, the two stability concepts coincide with each other.

4. EXTENDED MELAN THEOREM

The solid (or structure) B of Section 2 is here reconsidered and assumed to be D-stable for a wide class of load histories and initial conditions including the admissible ones.

The *fictitious elastic response* of B to some $\mathbf{P}$ combined with some initial plastic strains $\bar{\mathbf{E}}^{p}(\mathbf{X})$ is, by definition, the response of B computed by considering the material as elastic at any stress level, i.e. ignoring the existence of a yield function. Obviously, such a fictitious elastic response is an actual response only for stress levels below yield. For assigned values of $\mathbf{P}$ and $\bar{\mathbf{E}}^{p}$, the fictitious elastic response, denoted with cupped symbols as $\hat{\mathbf{u}}$, $\hat{\mathbf{E}}$, etc., is governed by the following equations, i.e.

$$\hat{\mathbf{F}} = \mathbf{I} + \nabla\hat{\mathbf{u}} \quad \text{in } V \tag{22a}$$

$$\hat{\mathbf{E}} = \tfrac{1}{2}(\hat{\mathbf{F}}^{T}\cdot\hat{\mathbf{F}} - \mathbf{I}) \quad \text{in } V \tag{22b}$$

$$\hat{\mathbf{u}} = \mathbf{0} \quad \text{on } \partial_D V \tag{22c}$$

$$\operatorname{div}(\hat{\mathbf{S}}\cdot\hat{\mathbf{F}}^{T}) + \mathbf{b} = \mathbf{0} \quad \text{in } V \tag{23a}$$

$$\mathbf{n}\cdot\hat{\mathbf{S}}\cdot\hat{\mathbf{F}}^{T} = \mathbf{f} \quad \text{on } \partial_T V \tag{23b}$$

$$\hat{\mathbf{S}} = \mathbf{C} : (\hat{\mathbf{E}} - \bar{\mathbf{E}}^{p}) \quad \text{in } V \tag{24}$$

which hold for arbitrary pairs $\mathbf{P}$ and $\bar{\mathbf{E}}^{p}(\mathbf{X})$.

According to the usual definition, (elastic) shakedown means that the structural response to whatsoever ALH is characterized by (limited) plastic flow only within an initial transient phase, after which it becomes elastic, that is no further plastic strains occur. For large displacements, the D-stability constitutes a structure's pre-requisite in order to establish *a priori* shakedown criteria. For a structure which is not D-stable the occurrence of shakedown is perhaps to be considered exceptional, if not impossible. The following can be proved.

Extended Melan's theorem

For a given D-stable elastic-perfectly plastic solid, or structure, subjected to loads variable in a given domain Π, a necessary and sufficient condition for (elastic) shakedown to occur is that there exist some initial plastic strains $\bar{\mathbf{E}}^p(\mathbf{X})$ such that the fictitious elastic stress response, $\hat{\mathbf{S}}$, to the loads $\mathbf{P} \in \Pi$ combined with this $\bar{\mathbf{E}}^p$ nowhere violates the yield condition, i.e.

$$\phi(\hat{\mathbf{S}}) \leqslant 0 \quad \text{in } V \times \Pi. \tag{25}$$

Proof of the necessity. Assume that shakedown occurs. Then, let $\mathbf{P}(t)$, $t \geqslant 0$, be any ALH and let $\mathbf{E}^p_s(\mathbf{X}) := \mathbf{E}^p(\mathbf{X}, t_s)$ be the actual plastic strains accumulated in the transient elastic–plastic phase $0 \leqslant t \leqslant t_s$ of the related equilibrium path $\mathbf{x}(\mathbf{X}, t)$. For $t > t_s$, the actual stress state of B, $\mathbf{S}$, is always below yield and can be expressed as the fictitious elastic response of B to the combined action of $\mathbf{P}(t)$ with $\mathbf{E}^p_s$ applied at $t = t_s$; that is, $\mathbf{S}$ coincides with $\hat{\mathbf{S}}$ of eqns (22)–(24) provided that $\bar{\mathbf{E}}^p$ is identified with $\mathbf{E}^p_s$. Since after t_s any load $\mathbf{P} \in \Pi$ is potentially active, it follows that eqn (25) is satisfied.

Proof of the sufficiency. Let $\hat{\mathbf{u}}$, $\hat{\mathbf{F}}$, $\hat{\mathbf{E}}$, $\hat{\mathbf{S}}$ be the fictitious elastic response of B to any load $\mathbf{P} \in \Pi$ combined with some $\bar{\mathbf{E}}^p$, and assume that correspondingly eqn (25) is satisfied. Also, let $\mathbf{u}$, $\mathbf{F}$, $\mathbf{E}$, $\mathbf{S}$ denote the actual elastic–plastic response of B to an arbitrarily chosen ALH. By the material stability postulate (Martin, 1975; Lubliner, 1990), one can write

$$(\mathbf{S} - \hat{\mathbf{S}}) : \dot{\mathbf{E}}^p \geqslant 0 \quad \text{in } V \times (0, T). \tag{26}$$

On setting $\Delta\mathbf{u} := \mathbf{u} - \hat{\mathbf{u}}$, $\Delta\mathbf{F} := \mathbf{F} - \hat{\mathbf{F}}$, $\Delta\mathbf{S} := \mathbf{S} - \hat{\mathbf{S}}$, etc. and making use of the identity

$$\dot{\mathbf{E}}^p = \Delta\dot{\mathbf{E}} - \mathbf{C}^{-1} : \Delta\dot{\mathbf{S}}, \tag{27}$$

eqn (26) can be rewritten, after an integration over V, as

$$\int_V \Delta\mathbf{S} : \dot{\mathbf{E}}^p \, \mathrm{d}V = \int_V \Delta\mathbf{S} : \Delta\dot{\mathbf{E}} \, \mathrm{d}V - \int_V \Delta\mathbf{S} : \mathbf{C}^{-1} : \Delta\dot{\mathbf{S}} \, \mathrm{d}V. \tag{28}$$

Then, by means of eqns (12a–c), and applying the virtual work principle (9), after some easy manipulations (not reported here for brevity), one obtains

$$\int_V \Delta\mathbf{S} : \dot{\mathbf{E}}^p \, \mathrm{d}V = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_V \Delta\mathbf{S} : \mathbf{C}^{-1} : \Delta\mathbf{S} \, \mathrm{d}V + \int_V \hat{\mathbf{S}} : \Delta^{(2)}\mathbf{E} \, \mathrm{d}V\right) + \int_V [(\Delta\mathbf{S} \cdot \Delta\mathbf{F}^T) : \dot{\hat{\mathbf{F}}} + \Delta^{(2)}\mathbf{E} : \dot{\hat{\mathbf{S}}}] \, \mathrm{d}V. \tag{29}$$

Note that, since $\hat{\mathbf{S}}$ does not violate the yield condition anywhere, the fictitious elastic response is also the actual response of B to the ALH combined with $\bar{\mathbf{E}}^p$; in addition, the assumed D-stability of the structure is sufficiently wide as to consider D-stable the equilibrium paths $\hat{\mathbf{x}}(\mathbf{X}, t)$ pertinent to any ALH combined with some $\bar{\mathbf{E}}^p$. Thus the D-stability functional related to $\hat{\mathbf{x}}(\mathbf{X}, t)$ as fundamental equilibrium path, $\hat{W}(t) = \hat{W}_F(t) + \hat{W}_G(t)$, is positive definite in some set of neighbor equilibrium paths, which includes the equilibrium paths $\mathbf{x}(\mathbf{X}, t)$ relative to all ALHs. [The neighbor equilibrium path $\mathbf{x}(\mathbf{X}, t)$ is obtained considering a perturbance that consists only in removing the initial plastic strains $\bar{\mathbf{E}}^p(\mathbf{X})$.] As $\hat{W}_F$ and $\hat{W}_G$, remembering eqns (17) and (18), read

$$\hat{W}_F(t) = \left(\frac{1}{2}\int_V \Delta\mathbf{S}:\mathbf{C}^{-1}:\Delta\mathbf{S}\,\mathrm{d}V + \int_V \hat{\mathbf{S}}:\Delta^{(2)}\mathbf{E}\,\mathrm{d}V\right)\Bigg|_t \tag{30}$$

and

$$\hat{W}_G(t) = -\int_0^t \left[\int_V (\Delta\mathbf{S}\cdot\Delta\mathbf{F}^{\mathrm{T}}):\dot{\hat{\mathbf{F}}}\,\mathrm{d}V + \int_V \Delta^{(2)}\mathbf{E}:\dot{\hat{\mathbf{S}}}\,\mathrm{d}V\right]\mathrm{d}\bar{t} \tag{31}$$

where $\bar{t}$ is an integration time variable, it follows that eqn (29) can be written as

$$\frac{\mathrm{d}\hat{W}}{\mathrm{d}t} = -\int_V \Delta\mathbf{S}:\dot{\mathbf{E}}^{\mathrm{p}}\,\mathrm{d}V \leqslant 0, \quad \forall t \geqslant 0. \tag{32}$$

A classical argument of shakedown theory is encountered here. Namely, since by eqn (32) $\hat{W}(t)$ must decrease as t increases, it follows that, whatever the ALH, a time t_s must arrive at which $\mathrm{d}\hat{W}/\mathrm{d}t = 0$, and this implies that eqn (26) is satisfied as an equality for $t > t_s$, a condition that can be satisfied if, and only if, $\dot{\mathbf{E}}^{\mathrm{p}} \equiv \mathbf{0}$ for all $t \geqslant t_s$; that is, if and only if, elastic shakedown occurs.

The boundedness of the total plastic work produced can also be shown using classical arguments (Koiter, 1960). In fact, assuming that inequality (25) is satisfied in the more stringent form $\phi(\mu\hat{\mathbf{S}}) \leqslant 0$ with $\mu > 1$ being a scalar, eqn (26) can be restated as

$$(\mathbf{S}-\mu\hat{\mathbf{S}}):\dot{\mathbf{E}}^{\mathrm{p}} \geqslant 0 \quad \text{in } V\times(0,T) \tag{33}$$

which is equivalent to

$$-(\mu-1)\mathbf{S}:\dot{\mathbf{E}}^{\mathrm{p}} + \mu\Delta\mathbf{S}:\dot{\mathbf{E}}^{\mathrm{p}} \leqslant 0 \quad \text{in } V\times(0,T). \tag{34}$$

On integration upon $V\times(0, T)$ and making use of eqn (32), eqn (34) gives

$$\int_0^T\int_V \mathbf{S}:\dot{\mathbf{E}}^{\mathrm{p}}\,\mathrm{d}V\,\mathrm{d}t \leqslant \frac{\mu}{\mu-1}[\hat{W}(0)-\hat{W}(T)] \tag{35}$$

which can be enforced by dropping the subtractive positive term, namely

$$\int_0^T\int_V \mathbf{S}:\dot{\mathbf{E}}^{\mathrm{p}}\,\mathrm{d}V\,\mathrm{d}t = \int_0^T\int_V D(\dot{\mathbf{E}}^{\mathrm{p}})\,\mathrm{d}V\,\mathrm{d}t \leqslant \frac{\mu}{\mu-1}\hat{W}(0). \tag{36}$$

As the latter inequality holds even for $T\to\infty$, and since $\hat{W}(0)$ is finite, inequality (36) proves that the overall plastic dissipation work produced in any ALH is bounded.

Remark 1. In case of small displacements, hence of zero geometrical effects, the fictitious elastic responses of B to $\mathbf{P}(t)$ and to $\bar{\mathbf{E}}^{\mathrm{p}}$ acting in sequence are no longer coupled with each other, and in particular the stress response to $\bar{\mathbf{E}}^{\mathrm{p}}$ turns out to be time-independent. Thus, the theorem above takes on the format of the classical Melan theorem for infinitesimal displacements (Koiter, 1963; Martin, 1975).

Remark 2. A scalar $m_s > 0$ is a *static* shakedown load multiplier if Melan's condition of eqn (25) is satisfied for loads $\mathbf{P} = m_s\bar{\mathbf{P}}$, $\bar{\mathbf{P}}\in\bar{\Pi}$, where $\bar{\Pi}$ is a reference domain. The supremum of m_s with respect to $\bar{\mathbf{E}}^{\mathrm{p}}$ and with respect to all ALHs, say m_s^*, is the structure's shakedown limit load multiplier (see Remark 5).

Remark 3. For small displacements, eqn (25) can be simplified considering, instead of all $\mathbf{P}\in\Pi$, only the loads $\mathbf{P}\in\Pi_{\mathrm{BL}}\subset\Pi$ where Π_{BL} is the set of the "basic loads", that is the smallest (even discrete) subset of Π whose convex hull coincides with the convex hull of Π (and with Π itself if Π is convex) (Polizzotto *et al.*, 1991). With large displacements, due to the geometrical nonlinearities, such a simplification is no longer possible, in general; however, simplifying procedures like the above may be adopted in practice according to the particular problem being studied. For instance, it is reasonable to conjecture that the computation of m_s^* of Remark 2 can be achieved considering a single ALH, that is the load path coincident with the boundary of the convex hull of Π.

Remark 4. It is evident from the proof of the extended Melan theorem (sufficiency part) that the property of the functional $\hat{W}(t)$, that was crucial to infer the shakedown occurrence, is that $\hat{W}(t)$ is *bounded from below*. The same occurs with other applications of the D-stability principle, e.g. to address the existence of a steady cycle (see Section 6). This circumstance enables one to regard the D-stability functional $W(t)$ of eqns (17)–(19) as being specified within an additive *finite* constant $\bar{W}$, i.e. $W(t) = W_F(t) + W_G(t) \geqslant \bar{W}$, $\forall t$.

5. EXTENDED KOITER THEOREM

A central concept of the kinematical approach to shakedown is Koiter's "kinematically admissible plastic strain cycle" (Koiter, 1960; Martin, 1975; König, 1987), or, equivalently, the so called *Plastic Accumulation Mechanism* (PAM) [see Polizzotto *et al.* (1991)]. A PAM is here defined as a plastic strain history, say $\mathbf{E}^{pc}(\mathbf{X}, t)$, $\mathbf{X} \in V$, $0 \leqslant t \leqslant T$, to be imposed upon B considered elastic and being in the deformed configuration $\hat{\mathbf{x}}(\mathbf{X}, t) = \mathbf{X} + \hat{\mathbf{u}}(\mathbf{X}, t)$ promoted by some ALH, say $\mathbf{P}(t)$, $0 \leqslant t \leqslant T$, acting together with some initial plastic strain field $\bar{\mathbf{E}}^p$. Without loss of generality, a cyclic ALH can be considered, i.e. $\mathbf{P}(0) = \mathbf{P}(T)$. The above plastic strain history is such that the cumulated strains $\mathbf{E}^{pc}(\mathbf{X}, T)$ are compatible in the body's final configuration $\hat{\mathbf{x}}(\mathbf{X}, T)$; or, equivalently, such that the fictitious elastic stress response increment, say $\mathbf{S}^c(\mathbf{X}, t)$, caused by the imposed strain history, vanishes at the final time, i.e. $\mathbf{S}^c(\mathbf{X}, T) = \mathbf{0}$ in V. In other words, the application of a PAM upon the body considered elastic and already loaded by a cyclic ALH and initial plastic strains produces, after a load cycle, no changes in the stress state existing at the beginning of the cycle. The above definition of PAM complies with the analogous definition for small displacements, the only difference being that, in case of small displacements, the fictitious elastic response of B to the imposed plastic strains $\mathbf{E}^{pc}(\mathbf{X}, t)$ can be computed considering B as being unloaded.

No magnitude limitations are in principle required for $\mathbf{E}^{pc}$; however $\mathbf{E}^{pc}$ and the rates $\dot{\mathbf{E}}^{pc}(\mathbf{X}, t)$ are assumed small at every t, which is justified by the consideration that $\mathbf{E}^{pc}$ is representative (see Section 6) of the incipient inadaptation collapse mode at the shakedown limit. The mathematical characterization of $\mathbf{E}^{pc}$ as a PAM is achieved through the equation set governing the fictitious elastic rate response of B to it with B being in the elastically deformed configuration $\hat{\mathbf{x}}(\mathbf{X}, t) = \mathbf{X} + \hat{\mathbf{u}}(\mathbf{X}, t)$ produced by some ALH combined with some $\bar{\mathbf{E}}^p(\mathbf{X})$. To this aim, taking account for the assumed smallness of $\dot{\mathbf{E}}^{pc}$ and since $\dot{\mathbf{F}}^c = \nabla\dot{\mathbf{u}}^c$, one can write the equations:

$$\dot{\mathbf{E}}^c = \tfrac{1}{2}[\hat{\mathbf{F}}^T \cdot \nabla\dot{\mathbf{u}}^c + (\nabla\dot{\mathbf{u}}^c)^T \cdot \hat{\mathbf{F}}] \quad \text{in } V \times (0, T) \tag{37a}$$

$$\dot{\mathbf{u}}^c = \mathbf{0} \quad \text{on } \partial_D V \times (0, T) \tag{37b}$$

$$\operatorname{div}(\dot{\mathbf{S}}^c \cdot \hat{\mathbf{F}}^T) + \operatorname{div}[\hat{\mathbf{S}} \cdot (\nabla\dot{\mathbf{u}}^c)^T] = \mathbf{0} \quad \text{in } V \times (0, T) \tag{38a}$$

$$\mathbf{n} \cdot \dot{\mathbf{S}}^c \cdot \hat{\mathbf{F}}^T + \mathbf{n} \cdot \hat{\mathbf{S}} \cdot (\nabla\dot{\mathbf{u}}^c)^T = \mathbf{0} \quad \text{on } \partial_T V \times (0, T) \tag{38b}$$

$$\dot{\mathbf{E}}^c = \mathbf{C}^{-1} : \dot{\mathbf{S}}^c + \dot{\mathbf{E}}^{pc} \quad \text{in } V \times (0, T) \tag{39}$$

$$\mathbf{S}^c(\mathbf{X}, T) = \int_0^T \dot{\mathbf{S}}^c(\mathbf{X}, t)\,\mathrm{d}t = \mathbf{0} \quad \text{in } V. \tag{40}$$

It can be observed that the total strain rates $\dot{\mathbf{E}}^c(\mathbf{X}, t)$, linearly expressed in terms of $\dot{\mathbf{u}}^c$ by eqn (37a), are compatible in the body's configuration specified by $\hat{\mathbf{u}}(\mathbf{X}, t)$; additionally, the equilibrium conditions for the stress rates $\dot{\mathbf{S}}^c(\mathbf{X}, t)$ are expressed in the same configuration $\hat{\mathbf{u}}(\mathbf{X}, t)$ [i.e. the geometric effects caused by the time variability of $\hat{\mathbf{u}}$ are disregarded due to the smallness of $\dot{\mathbf{u}}^c$, eqns (38a,b)].

PAM is any field $\dot{\mathbf{E}}^{pc}(\mathbf{X}, t)$ specified in $V\times(0, T)$ which together with the related increment variables $(\cdot)^c$ satisfy eqns (37)–(40) for some ALH and initial plastic strains. The set of all such PAMs for assigned ALH and initial plastic strains is denoted M.

A PAM can be shown to possess the following two properties:

(1) The cumulated imposed strains at the final time T, $\mathbf{E}^{pc}(\mathbf{X}, T)$, are compatible with the displacement increments $\mathbf{u}^c(\mathbf{X}, T)$ upon the final configuration $\hat{\mathbf{x}}(\mathbf{X}, T) = \mathbf{X}+\hat{\mathbf{u}}(\mathbf{X}, T)$ of B, that is

$$\mathbf{E}^{pc}|_{t=T} = \tfrac{1}{2}[\hat{\mathbf{F}}^T\cdot\nabla\mathbf{u}^c+(\nabla\mathbf{u}^c)^T\cdot\hat{\mathbf{F}}+(\nabla\mathbf{u}^c)^T\cdot\nabla\mathbf{u}^c]_{t=T} \tag{41a}$$

$$\simeq \tfrac{1}{2}[\hat{\mathbf{F}}^T\cdot\nabla\mathbf{u}^c+(\nabla\mathbf{u}^c)^T\cdot\hat{\mathbf{F}}]_{t=T} \quad \text{in } V \tag{41b}$$

where the term $(\nabla\mathbf{u}^c)^T\cdot\nabla\mathbf{u}^c/2$ has been dropped in consideration of the assumed smallness of $\mathbf{E}^{pc}$, hence of $\mathbf{u}^c$. To prove that, let one note that $\dot{\mathbf{E}}^c$, as a strain increment, is to be compared with $\Delta\dot{\mathbf{E}}$ of eqn (12a) and it can thus be split as $\dot{\mathbf{E}}^c = \dot{\mathbf{E}}^c_{(1)}+\dot{\mathbf{E}}^c_{(2)}$ with $\dot{\mathbf{E}}^c_{(1)}$ and $\dot{\mathbf{E}}^c_{(2)}$ given by eqns (12b,c), respectively, but here $\dot{\mathbf{u}}^c$ and $\hat{\mathbf{u}}$ are to be used in place of $\Delta\mathbf{u}$ and $\mathbf{u}^{(a)}$ of eqns (12b,c). With these changes, it can be recognized that $\dot{\mathbf{E}}^c_{(1)}$ identifies with the right-hand side of eqn (37a) and that therefore $\dot{\mathbf{E}}^c_{(2)}$ is identically vanishing. Next, an integration of eqn (37a) over $(0, T)$ gives

$$\mathbf{E}^{pc}|_{t=T} = \frac{1}{2}\int_0^T [\nabla\dot{\mathbf{u}}^c+(\nabla\dot{\mathbf{u}}^c)^T+(\nabla\hat{\mathbf{u}})^T\cdot(\nabla\dot{\mathbf{u}}^c)+(\nabla\dot{\mathbf{u}}^c)^T\cdot(\nabla\hat{\mathbf{u}})]\,dt$$

$$= \tfrac{1}{2}[\nabla\mathbf{u}^c+(\nabla\mathbf{u}^c)^T+(\nabla\hat{\mathbf{u}})^T\cdot\nabla\mathbf{u}^c+(\nabla\mathbf{u}^c)^T\cdot\nabla\hat{\mathbf{u}}+(\nabla\mathbf{u}^c)^T\cdot\nabla\mathbf{u}^c]|_{t=T}-\int_0^T \dot{\mathbf{E}}^c_{(2)}\,dt, \square \tag{42}$$

where $\dot{\mathbf{E}}^c_{(2)}$ has the expression:

$$\dot{\mathbf{E}}^c_{(2)} = \tfrac{1}{2}[(\nabla\mathbf{u}^c)^T\cdot\nabla\dot{\mathbf{u}}^c+(\nabla\dot{\mathbf{u}}^c)^T\cdot\nabla\mathbf{u}^c+(\nabla\dot{\hat{\mathbf{u}}})^T\cdot\nabla\mathbf{u}^c+(\nabla\mathbf{u}^c)^T\cdot\nabla\dot{\hat{\mathbf{u}}}]. \tag{43}$$

This expression is analogous to eqn (12c), but with $\mathbf{u}^c$ and $\hat{\mathbf{u}}$ in place of $\Delta\mathbf{u}$ and $\mathbf{u}^{(a)}$, respectively. By eqn (37a), which states that $\dot{\mathbf{E}}^c = \dot{\mathbf{E}}^{c(1)}$ and hence $\dot{\mathbf{E}}^{c(2)} = \mathbf{0}$ identically, it can be stated that eqn (42) coincides—to within a simple formal transformation—with eqn (41b).

(2) The fictitious volume and surface forces that simulate the geometric effects due to the PAM, accumulated at the final time T, are identically vanishing, namely

$$\int_0^T \operatorname{div}[\dot{\mathbf{S}}^c\cdot(\nabla\hat{\mathbf{u}})^T+\hat{\mathbf{S}}\cdot(\nabla\dot{\mathbf{u}}^c)^T]\,dt = \mathbf{0} \quad \text{in } V \tag{44a}$$

$$\int_0^T \mathbf{n}\cdot[\dot{\mathbf{S}}^c\cdot(\nabla\hat{\mathbf{u}})^T+\hat{\mathbf{S}}\cdot(\nabla\dot{\mathbf{u}}^c)^T]\,dt = \mathbf{0} \quad \text{on } \partial_T V. \tag{44b}$$

These equations are easily derived from eqns (38a,b) upon integration over $(0, T)$ and with consideration of eqn (40).

The total work L^c_e performed by the actual volume and surface forces acting on B through the fictitious elastic displacement rates $\dot{\mathbf{u}}^c$ produced by the PAM in a complete cycle is

$$L_{\mathrm{e}}^{\mathrm{c}} := \int_0^T \left(\int_V \mathbf{b} \cdot \dot{\mathbf{u}}^{\mathrm{c}} \, \mathrm{d}V + \int_{\partial_T V} \mathbf{f} \cdot \dot{\mathbf{u}}^{\mathrm{c}} \, \mathrm{d}S \right) \mathrm{d}t. \tag{45}$$

Applying the virtual work principle (9), $L_{\mathrm{e}}^{\mathrm{c}}$ can also be written

$$L_{\mathrm{e}}^{\mathrm{c}} = \int_0^T \int_V \hat{\mathbf{S}} : \dot{\mathbf{E}}^{\mathrm{c}} \, \mathrm{d}V \, \mathrm{d}t \tag{46}$$

(note that $\hat{\mathbf{S}}$ is in equilibrium with $\mathbf{b}$ and $\mathbf{f}$ in the configuration $\hat{\mathbf{u}}$, and that $\dot{\mathbf{E}}^{\mathrm{c}} = \dot{\mathbf{E}}^{\mathrm{c}}_{(1)}$ is compatible with $\dot{\mathbf{u}}^{\mathrm{c}}$ in the same configuration $\hat{\mathbf{u}}$). With the aid of eqns (37)–(40) and (22)–(24), eqn (46) can then be transformed as

$$\begin{aligned} L_{\mathrm{e}}^{\mathrm{c}} &= \int_0^T \int_V \hat{\mathbf{S}} : \dot{\mathbf{E}}^{\mathrm{pc}} \, \mathrm{d}V \, \mathrm{d}t + \int_0^T \int_V \hat{\mathbf{S}} : \mathbf{C}^{-1} : \dot{\mathbf{S}}^{\mathrm{c}} \, \mathrm{d}V \, \mathrm{d}t \\ &= \int_0^T \int_V \hat{\mathbf{S}} : \dot{\mathbf{E}}^{\mathrm{pc}} \, \mathrm{d}V \, \mathrm{d}t + \int_0^T \int_V (\hat{\mathbf{E}}^{(1)} + \hat{\mathbf{E}}^{(2)} - \bar{\mathbf{E}}^{\mathrm{p}}) : \dot{\mathbf{S}}^{\mathrm{c}} \, \mathrm{d}V \, \mathrm{d}t. \end{aligned} \tag{47}$$

Applying the divergence theorem and with the aid of eqns (38a,b), it is

$$\begin{aligned} \int_V \hat{\mathbf{E}}^{(1)} : \dot{\mathbf{S}}^{\mathrm{c}} \, \mathrm{d}V &= -\int_V \operatorname{div} \dot{\mathbf{S}}^{\mathrm{c}} \cdot \hat{\mathbf{u}} \, \mathrm{d}V + \int_{\partial_T V} \mathbf{n} \cdot \dot{\mathbf{S}}^{\mathrm{c}} \cdot \hat{\mathbf{u}} \, \mathrm{d}S \\ &= \int_V \operatorname{div} [\dot{\mathbf{S}}^{\mathrm{c}} \cdot (\nabla \hat{\mathbf{u}})^{\mathrm{T}} + \hat{\mathbf{S}} \cdot (\nabla \dot{\mathbf{u}}^{\mathrm{c}})^{\mathrm{T}}] \cdot \hat{\mathbf{u}} \, \mathrm{d}V \\ &\quad - \int_{\partial_T V} \mathbf{n} \cdot [\dot{\mathbf{S}}^{\mathrm{c}} \cdot (\nabla \hat{\mathbf{u}})^{\mathrm{T}} + \hat{\mathbf{S}} \cdot (\nabla \dot{\mathbf{u}}^{\mathrm{c}})^{\mathrm{T}}] \cdot \hat{\mathbf{u}} \, \mathrm{d}S \end{aligned} \tag{48}$$

and then, applying again the divergence theorem,

$$\begin{aligned} \int_V \hat{\mathbf{E}}^{(1)} : \dot{\mathbf{S}}^{\mathrm{c}} \, \mathrm{d}V &= -\int_V [\dot{\mathbf{S}}^{\mathrm{c}} \cdot (\nabla \hat{\mathbf{u}})^{\mathrm{T}}] : \nabla \hat{\mathbf{u}} \, \mathrm{d}V - \int_V [\hat{\mathbf{S}} \cdot (\nabla \dot{\mathbf{u}}^{\mathrm{c}})^{\mathrm{T}}] : \nabla \hat{\mathbf{u}} \, \mathrm{d}V \\ &= -\int_V \dot{\mathbf{S}}^{\mathrm{c}} : [(\nabla \hat{\mathbf{u}})^{\mathrm{T}} \cdot \nabla \hat{\mathbf{u}}] \, \mathrm{d}V - \int_V [\hat{\mathbf{S}} \cdot (\nabla \hat{\mathbf{u}})^{\mathrm{T}}] : \nabla \dot{\mathbf{u}}^{\mathrm{c}} \, \mathrm{d}V \\ &= -2 \int_V \dot{\mathbf{S}}^{\mathrm{c}} : \hat{\mathbf{E}}^{(2)} \, \mathrm{d}V - \int_V [\hat{\mathbf{S}} \cdot (\nabla \hat{\mathbf{u}})^{\mathrm{T}}] : \nabla \dot{\mathbf{u}}^{\mathrm{c}} \, \mathrm{d}V. \end{aligned} \tag{49}$$

Taking into account eqn (40), eqn (47) finally becomes

$$L_{\mathrm{e}}^{\mathrm{c}} = \int_0^T \int_V \hat{\mathbf{S}} : \dot{\mathbf{E}}^{\mathrm{pc}} \, \mathrm{d}V \, \mathrm{d}t - \int_0^T \int_V [\hat{\mathbf{S}} \cdot (\nabla \hat{\mathbf{u}})^{\mathrm{T}}] : \nabla \dot{\mathbf{u}}^{\mathrm{c}} \, \mathrm{d}V \, \mathrm{d}t - \int_0^T \int_V \hat{\mathbf{E}}^{(2)} : \dot{\mathbf{S}}^{\mathrm{c}} \, \mathrm{d}V \, \mathrm{d}t. \tag{50}$$

The first subtractive integral term on the right-hand side of eqn (50) represents the total work performed, through $\dot{\mathbf{u}}^{\mathrm{c}}$, by the fictitious volume and surface forces simulating the geometry change effects in the deformed configuration $\hat{\mathbf{x}}(\mathbf{X}, t)$; that is the forces

$$\begin{aligned} \hat{\mathbf{b}}_G &:= \operatorname{div} [\hat{\mathbf{S}} \cdot (\nabla \hat{\mathbf{u}})^{\mathrm{T}}] \quad \text{in } V \times (0, T), \\ \hat{\mathbf{f}}_G &:= \mathbf{n} \cdot \hat{\mathbf{S}} \cdot (\nabla \hat{\mathbf{u}})^{\mathrm{T}} \quad \text{on } \partial_T V \times (0, T). \end{aligned} \tag{51}$$

Moreover, the second subtractive term on the right-hand side of eqn (50) is the work done, through $\dot{\mathbf{S}}^c$, by the second-order part of $\hat{\mathbf{E}}$, i.e. $\hat{\mathbf{E}}^{(2)} = (\nabla\hat{\mathbf{u}})^T \cdot \nabla\hat{\mathbf{u}}/2$.

After the above preliminaries, the following can be stated.

Extended Koiter's theorem.

For a given D-stable elastic–perfectly-plastic solid, or structure, subjected to loads $\mathbf{P}$ variable within a given domain Π, a necessary and sufficient condition for shakedown not to occur is that there exists an ALH for which, whatever the initial plastic strains $\bar{\mathbf{E}}^p(\mathbf{X})$ associated with it, a PAM can be found such as to satisfy the following energy inequality, i.e.

$$K[\dot{\mathbf{E}}^{pc}] := \int_0^T \int_V D(\dot{\mathbf{E}}^{pc})\,\mathrm{d}V\,\mathrm{d}t - \int_0^T \int_V \{[\hat{\mathbf{S}} \cdot (\nabla\hat{\mathbf{u}})^T] : \nabla\dot{\mathbf{u}}^c + \hat{\mathbf{E}}^{(2)} : \dot{\mathbf{S}}^c\}\,\mathrm{d}V\,\mathrm{d}t - L_e^c < 0. \quad (52)$$

Proof of the sufficiency. By hypothesis, for some ALH, say $\mathbf{P}^*(t)$, $0 \leqslant t \leqslant T$, and for arbitrary $\bar{\mathbf{E}}^p(\mathbf{X})$, there exists a PAM, $\dot{\mathbf{E}}^{pc}(\mathbf{X}, t)$, such as to satisfy eqn (52). By absurdity, assume that correspondingly elastic shakedown occurs. Then, in virtue of the assumed D-stability, Melan's theorem of Section 4 can be applied to state that an initial plastic strain field, say $\bar{\mathbf{E}}^{p*}$, can be found such that the fictitious elastic stress response, $\hat{\mathbf{S}}^*$, of B to $\bar{\mathbf{E}}^{p*}$ combined with an arbitrary ALH is plastically admissible, i.e. $\phi(\hat{\mathbf{S}}^*) \leqslant 0$ in $V \times (0, T)$. Since the latter plasticity condition holds good also if $\bar{\mathbf{E}}^{p*}$ is combined with $\mathbf{P}^*(t)$, by virtue of the maximum plastic work theorem one has

$$D(\dot{\mathbf{E}}^{pc}) \geqslant \hat{\mathbf{S}}^* : \dot{\mathbf{E}}^{pc} \quad \text{in } V \times (0, T). \quad (53)$$

By eqns (39), (22) and (24) one can write

$$\hat{\mathbf{S}}^* : \dot{\mathbf{E}}^{pc} = \hat{\mathbf{S}}^* : (\dot{\mathbf{E}}^c - \mathbf{C}^{-1} : \dot{\mathbf{S}}^c) = \hat{\mathbf{S}}^* : \dot{\mathbf{E}}^c - (\hat{\mathbf{E}}^* - \bar{\mathbf{E}}^{p*}) : \dot{\mathbf{S}}^c, \quad (54)$$

hence, with an integration over $V \times (0, T)$,

$$\int_0^T \int_V \hat{\mathbf{S}}^* : \dot{\mathbf{E}}^{pc}\,\mathrm{d}V\,\mathrm{d}t = \int_0^T \int_V \hat{\mathbf{S}}^* : \dot{\mathbf{E}}^c\,\mathrm{d}V\,\mathrm{d}t - \int_0^T \int_V (\hat{\mathbf{E}}^{*(1)} + \hat{\mathbf{E}}^{*(2)} - \bar{\mathbf{E}}^{p*}) : \dot{\mathbf{S}}^c\,\mathrm{d}V\,\mathrm{d}t. \quad (55)$$

Making use of eqns (46) and (49), and in virtue of eqn (40), eqn (55) becomes

$$\int_0^T \int_V \hat{\mathbf{S}}^* : \dot{\mathbf{E}}^{pc}\,\mathrm{d}V\,\mathrm{d}t = L_e^{c*} + \int_0^T \int_V \{[\hat{\mathbf{S}}^* \cdot (\nabla\hat{\mathbf{u}}^*)^T] : \nabla\dot{\mathbf{u}}^c + \hat{\mathbf{E}}^{*(2)} : \dot{\mathbf{S}}^c\}\,\mathrm{d}V\,\mathrm{d}t. \quad (56)$$

Thus, on integration of eqn (53) over $V \times (0, T)$ and substituting from eqn (56), one can write the inequality

$$\int_0^T \int_V D(\dot{\mathbf{E}}^{pc})\,\mathrm{d}V\,\mathrm{d}t \geqslant L_e^{c*} + \int_0^T \int_V \{[\hat{\mathbf{S}}^* \cdot (\nabla\hat{\mathbf{u}}^*)^T] : \nabla\dot{\mathbf{u}}^c + \hat{\mathbf{E}}^{*(2)} : \dot{\mathbf{S}}^c\}\,\mathrm{d}V\,\mathrm{d}t. \quad (57)$$

This inequality is equivalent to $K[\dot{\mathbf{E}}^{pc}] \geqslant 0$, that is inequality (52) is violated, contrary to the initial hypothesis. Therefore, under validity of this initial hypothesis, shakedown cannot occur.

Proof of the necessity. By hypothesis, elastic shakedown does not occur. Let, by absurdity, eqn (52) be negated, i.e.

$$K[\dot{\mathbf{E}}^{pc}] \geqslant 0 \quad \forall \dot{\mathbf{E}}^{pc} \in M \tag{58}$$

and for all ALHs and some $\bar{\mathbf{E}}^p(\mathbf{X})$. Since M includes the trivial PAM $\dot{\mathbf{E}}^{pc} \equiv \mathbf{0}$, and $K[\mathbf{0}] = 0$, eqn (58) implies that the minimization problem

$$\min_{(\dot{\mathbf{E}}^{pc})} K[\dot{\mathbf{E}}^{pc}] \quad \text{subject to } \dot{\mathbf{E}}^{pc} \in M \tag{59}$$

admits, for every arbitrarily fixed ALH and for some $\bar{\mathbf{E}}^p$, an absolute (vanishing) minimum, and that therefore the Euler–Lagrange equation set related to eqn (59) must also admit a solution. On appending the constraint eqns (37)–(40) to the objective functional K of eqn (59), the augmented functional $\tilde{K}$ is obtained as

$$\begin{aligned}
\tilde{K} := K[\dot{\mathbf{E}}^{pc}] &+ \int_0^T \int_V \mathbf{S}^* : [\dot{\mathbf{E}}^c - \mathbf{C}^{-1} : \dot{\mathbf{S}}^c - \dot{\mathbf{E}}^{pc}]\, \mathrm{d}V\, \mathrm{d}t \\
&- \int_0^T \int_V \mathbf{S}^* : [\dot{\mathbf{E}}^c - \nabla \dot{\mathbf{u}}^c - \nabla \hat{\mathbf{u}}^{\mathrm{T}} \cdot \nabla \dot{\mathbf{u}}^c]\, \mathrm{d}V\, \mathrm{d}t \\
&- \int_0^T \int_V \operatorname{div} \{\dot{\mathbf{S}}^c \cdot [\mathbf{I} + (\nabla \hat{\mathbf{u}})^{\mathrm{T}}] + \hat{\mathbf{S}} \cdot (\nabla \dot{\mathbf{u}}^c)^{\mathrm{T}}\} \cdot \mathbf{u}^*\, \mathrm{d}V\, \mathrm{d}t \\
&+ \int_0^T \int_{\partial_T V} \mathbf{n} \cdot \{\dot{\mathbf{S}}^c \cdot [\mathbf{I} + (\nabla \hat{\mathbf{u}})^{\mathrm{T}}] + \hat{\mathbf{S}} \cdot (\nabla \dot{\mathbf{u}}^c)^{\mathrm{T}}\} \cdot \mathbf{u}^*\, \mathrm{d}S\, \mathrm{d}t \\
&- \int_V \bar{\mathbf{E}}^{p*} : \int_0^T \dot{\mathbf{S}}^c\, \mathrm{d}V\, \mathrm{d}t,
\end{aligned} \tag{60}$$

where $\mathbf{u}^*(\mathbf{X}, t)$, $\mathbf{S}^*(\mathbf{X}, t)$, and $\bar{\mathbf{E}}^{p*}$ $(\mathbf{X})$ are unknown (sufficiently regular) Lagrange multipliers ($\mathbf{S}^*$ and $\bar{\mathbf{E}}^{p*}$ being second-order symmetric tensors).

On taking the first variation of $\tilde{K}$ with respect to the variables $(\cdot)^c$ and $(\cdot)^*$ and with the aid of some mathematics (whose details are dropped for brevity), one obtains an equation system including, besides eqns (37)–(40), the following ones:

$$\tfrac{1}{2}[\nabla \mathbf{u}^* + (\nabla \mathbf{u}^*)^{\mathrm{T}} + (\nabla \hat{\mathbf{u}})^{\mathrm{T}} \cdot \nabla \hat{\mathbf{u}}] = \mathbf{C}^{-1} : \mathbf{S}^* + \bar{\mathbf{E}}^{p*} \quad \text{in } V \times (0, T) \tag{61a}$$

$$\mathbf{u}^* = \mathbf{0} \quad \text{on } \partial_D V \times (0, T) \tag{61b}$$

$$\operatorname{div}(\mathbf{S}^* \cdot \hat{\mathbf{F}}^{\mathrm{T}}) + \mathbf{b} = \mathbf{0} \quad \text{in } V \times (0, T) \tag{62a}$$

$$\mathbf{n} \cdot \mathbf{S}^* \cdot \hat{\mathbf{F}}^{\mathrm{T}} = \mathbf{f} \quad \text{on } \partial_T V \times (0, T) \tag{62b}$$

$$\mathbf{S}^* = \partial D / \partial \dot{\mathbf{E}}^{pc} \quad \text{in } V \times (0, T) \tag{63}$$

$$\operatorname{div} [\hat{\mathbf{S}} \cdot (\nabla \mathbf{u}^* - \nabla \hat{\mathbf{u}})^{\mathrm{T}}] = \mathbf{0} \quad \text{in } V \times (0, T) \tag{64a}$$

$$\mathbf{n} \cdot \hat{\mathbf{S}} \cdot (\nabla \mathbf{u}^* - \nabla \hat{\mathbf{u}})^{\mathrm{T}} = \mathbf{0} \quad \text{on } \partial V_T \times (0, T). \tag{64b}$$

The latter equation set enables one to disclose the meaning of the Lagrange multipliers. Equations (64a,b), since $\mathbf{u}^* = \hat{\mathbf{u}} = \mathbf{0}$ on $\partial_D V \times (0, T)$, imply that the fields $\mathbf{u}^*$ and $\hat{\mathbf{u}}$ coincide with each other. Therefore, by comparison of eqns (61)–(64) with eqns (22)–(24), it can be stated that $\mathbf{S}^*$ is the fictitious elastic stress response to the chosen ALH combined with the initial plastic strain $\bar{\mathbf{E}}^{p*}$, and that $\mathbf{S}^*$, being derived from the dissipation function by eqn (63), is plastically admissible, i.e. $\phi(\mathbf{S}^*) \leqslant 0$ in $V \times (0, T)$. As these results hold for all ALHs and for some initial plastic strain, and in consideration of the assumed D-stability pre-requisite for B, it follows that the extended Melan theorem applies and thus shakedown occurs. As this is impossible by virtue of the initial hypothesis, it finally follows that, if shakedown does not occur, there must exist some PAM such that—for some ALH and for all $\bar{\mathbf{E}}^{p}$—eqn (52) is satisfied [hence problem (59) has no solution].

The above Koiter's theorem, contrary to the classical one, is not of pure kinematic nature because of the presence of $\dot{\mathbf{S}}^{c}$ in eqn (52). It can be given an alternative form by simply negating the conditions for its validity. To this purpose, let K of eqn (52) be redefined as

$$K := L_{i}^{c} - L_{e}^{c} \tag{65}$$

where L_{i}^{c} collects the terms on the right-hand side of eqn (52), except the last one. Then the following theorem holds.

Extended Koiter's theorem—alternative form

For a given D-stable elastic–perfectly-plastic solid, or structure, a necessary and sufficient condition for shakedown to occur is that the inequality

$$K[\dot{\mathbf{E}}^{pc}] := L_{i}^{c}[\dot{\mathbf{E}}^{pc}] - L_{e}^{c}[\dot{\mathbf{E}}^{pc}] \geqslant 0, \quad \forall\, \dot{\mathbf{E}}^{pc} \in M \tag{66}$$

is satisfied for all ALHs and some $\bar{\mathbf{E}}^{p}$.

Remark 5. In case of small displacements, hence of zero geometrical effects, those terms in eqns (37)–(40) and on the right-hand side of eqn (52), which are coupled with the fictitious elastic response of B, drop and the theorem above takes on the classical format of Koiter's theorem for infinitesimal displacements [see Polizzotto *et al.* (1991)].

Remark 6. On denoting the loads as $\mathbf{P} = m\bar{\mathbf{P}}$, with $\bar{\mathbf{P}}$ being the reference load, a scalar $m_{k} > 0$ is a *kinematic* shakedown load multiplier if it is obtained as the ratio $m_{k} = L_{i}^{c}/\bar{L}_{e}^{c}$ where $\bar{L}_{e}^{c}$ is the work of the loads $\bar{\mathbf{P}}$ through the PAM. The infimum of m_{k}, say m_{k}^{*}, with respect to $\dot{\mathbf{E}}^{pc} \in M$ and with respect to all ALHs and $\bar{\mathbf{E}}^{p}$, constitutes the structure's shakedown limit load multiplier. The fact that $m_{k}^{*} = m_{s}^{*}$ can be proved as follows. Namely, for any load m_{s} shakedown occurs, hence $K[\dot{\mathbf{E}}^{pc}] = L_{i}^{c} - m_{s}\bar{L}_{e}^{c} \geqslant 0$, i.e. $m_{s} \leqslant L_{i}^{c}/\bar{L}_{e}^{c}$, $\forall \dot{\mathbf{E}}^{pc} \in M$, for all ALHs and for some $\bar{\mathbf{E}}^{p}$, and in particular $m_{s}^{*} \leqslant m_{k}^{*}$; on the other hand if m_{k} meets eqn (66), i.e. $L_{i}^{c} - m_{k}\bar{L}_{e}^{c} \geqslant 0$, $\forall \dot{\mathbf{E}}^{pc} \in M$, for all ALHs and for some $\bar{\mathbf{E}}^{p}$, shakedown occurs, hence $m_{k} \leqslant m_{s}^{*}$ and in particular it is $m_{k}^{*} \leqslant m_{s}^{*}$; that is $m_{k}^{*} = m_{s}^{*}$.

Remark 7. For small displacements, the definition of PAM can be greatly simplified because geometrical effects are no longer present and the body can be considered as unloaded; that is, eqns (37)–(40) transform by setting $\hat{\mathbf{F}} = \mathbf{I}$ and $\hat{\mathbf{S}} = \mathbf{0}$. Another simplification, parallel to that of Remark 3, can be achieved by defining $\dot{\mathbf{E}}^{pc} = \dot{\mathbf{E}}^{pc}(\mathbf{X}, \mathbf{P})$ with $\mathbf{P}$ ranging in Π_{BL}. This implies that time integrations over $(0, T)$ in this section are replaced by domain integrations over Π_{BL} (or simply by sums if Π_{BL} is discrete) [see Polizzotto *et al.* (1991)]. Such simplifications are no longer possible, in general, for large displacements; however, problem dependent simplifying procedures may be adopted in practice, for instance the computation of m_{k}^{*} of Remark 6 may perhaps be achieved considering only the boundary loads of the convex hull of Π (or Π if Π is convex).

6. EXISTENCE OF A STEADY CYCLE

The elastic–plastic body B of Section 2 is here considered subjected to a load $\mathbf{P}(\tau)$, $0 \leqslant \tau \leqslant \Delta t$, acting in subsequent cycles $n = 1, 2, 3, \ldots$ where τ is a local time variable. B is D-stable in a sufficiently wide range of load histories and initial conditions. On denoting the general time by $t = \tau + t_{n-1}$ with $t_{n} := n\Delta t$, let one observe that the segment of the actual equilibrium path relative to the nth cycle, i.e. $\mathbf{x}(\mathbf{X}, \tau + t_{n-1})$, $0 \leqslant \tau \leqslant \Delta t$, can be considered as the response of B to the loads combined with the initial plastic strains $\mathbf{E}^p_{n-1}(\mathbf{X})$, i.e. the plastic strains accumulated at the end of the previous $n-1$ cycles. The equilibrium path segment relative to the first cycle $\mathbf{x}(\mathbf{X}, \tau)$ is D-stable, by hypothesis; the equilibrium path segment relative to the nth cycle $\mathbf{x}(\mathbf{X}, \tau+t_{n-1})$ is also D-stable as long as $\mathbf{E}^p_{n-1}(\mathbf{X})$ does not exceed the structure's D-stability limits (note that $\mathbf{E}^p_{n-1}$ has the role of an initial imperfection). If $\mathbf{E}^p_n(\mathbf{X})$ increases in magnitude as n increases, loss of D-stability may occur at some subsequent (critical) cycle, N_c say. Qualitatively speaking, the greater the destabilizing effects caused by geometry changes, the smaller N_c; but $N_c = \infty$ if the geometry changes produce no destabilizing effects.

In case that N_c is infinite (but in practice only sufficiently large), it can be shown that—just like in the framework of small displacements [see, e.g. Martin (1975)]—the structural response tends to stabilize into a steady-state response (or steady cycle) characterized by periodic stresses and plastic strain rates.

To this purpose, let $\mathbf{u}(\mathbf{x}, t)$, $\mathbf{S}(\mathbf{x}, t)$, $\mathbf{E}(\mathbf{x}, t)$, etc., denote the response variables related to the (full-length) equilibrium path $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, where t is still the general time; also, let the symbol $\Delta(\cdot)$ denote the difference of the quantity $(\cdot)$ evaluated at times $t+\Delta t$ and t, for instance $\Delta\mathbf{S}(\mathbf{X}, t) := \mathbf{S}(\mathbf{X}, t+\Delta t)-\mathbf{S}(\mathbf{X}, t)$. By the material stability postulate, one can write

$$\Delta\mathbf{S}:\Delta\dot{\mathbf{E}}^p \geqslant 0 \quad \text{in } V, \quad \forall t \geqslant 0. \tag{67}$$

Since $\Delta\dot{\mathbf{E}}^p = \Delta\dot{\mathbf{E}} - \mathbf{C}^{-1}:\Delta\dot{\mathbf{S}}$, and since an equation formally identical to eqn (28) can be shown to hold, eqns (12a–c) and the virtual work principle (9) can be used in a procedure similar to that employed in Section 4 to derive eqn (29) from eqn (28). In this way one obtains the inequality

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\int_V \Delta\mathbf{S}:\Delta\dot{\mathbf{E}}^p\,\mathrm{d}V \leqslant 0, \quad \forall t \geqslant 0, \tag{68}$$

where the nonnegativity sign is a consequence of eqn (67) and $W(t)$ is given by

$$W(t) = W_F(t) + W_G(t) \tag{69}$$

where

$$W_F(t) = \left(\frac{1}{2}\int_V \Delta\mathbf{S}:\mathbf{C}^{-1}:\Delta\mathbf{S}\,\mathrm{d}V + \int_V \mathbf{S}:\Delta^{(2)}\mathbf{E}\,\mathrm{d}V\right)\Bigg|_t \tag{70a}$$

$$W_G(t) = -\int_0^t \left[\int_V (\Delta\mathbf{S}\cdot\Delta\mathbf{F}^T):\dot{\mathbf{F}}\,\mathrm{d}V + \int_V \Delta^{(2)}\mathbf{E}:\dot{\mathbf{S}}\,\mathrm{d}V\right]\mathrm{d}\bar{t}. \tag{70b}$$

Namely, $W(t)$ turns out to be the D-stability functional of the fundamental equilibrium path $\mathbf{x}(\mathbf{X}, t)$, $t \geqslant 0$, coupled with the neighbor equilibrium path $\mathbf{x}(\mathbf{X}, t+\Delta t)$, $t \geqslant 0$, the latter being obtained from the fundamental one through a shift of amplitude Δt towards the positive time axis. According to eqn (68), which is similar to eqn (32), $W(t)$ turns out to be a monotonically decreasing function of t as long as $\Delta\dot{\mathbf{E}}^p \neq \mathbf{0}$ even in a small portion of V; on the other hand, since $W(t) > 0$ for all $t > 0$ in the assumed hypothesis $N_c = \infty$, and it thus cannot take negative values, a time t_s must arrive at which $W(t)$ stops decreasing; that is, for $t \geqslant t_s$, it is $W(t) =$ constant, which implies that eqn (67) is satisfied as an equality for $t \geqslant t_s$. The latter condition, in turn, implies, according to known results of plasticity theory [see, e.g. Martin (1975)], that $\Delta\dot{\mathbf{E}}^p = \mathbf{0}$ everywhere in V for all $t \geqslant t_s$, and that $\Delta\mathbf{S} = \mathbf{0}$ in $V_p \subset V$ and for all $t \geqslant t_s$ in which plastic yielding occurs. Observing that $\dot{\mathbf{S}}$ can be regarded as the fictitious elastic stress rate response history induced in B by the (periodic) load rate $\dot{\mathbf{P}}(t)$ together with an imposed strain rate history coincident with $\dot{\mathbf{E}}^p(\mathbf{X}, t)$—which is periodic for $t \geqslant t_s$ everywhere in V—whereas the initial conditions are taken as the body's state at

t_s, it follows that $\dot{\mathbf{S}}$ is periodic like $\dot{\mathbf{E}}^p$, hence $\Delta\dot{\mathbf{S}} = \mathbf{0}$ in V for all $t \geqslant t_s$; that is, $\Delta\mathbf{S}$ is time-independent in V for $t \geqslant t_s$. Since, as a consequence of what has previously been established, $\Delta\mathbf{S}$ vanishes in V_p—the region where plastic yielding occurs after t_s—it must necessarily vanish in the whole V. In conclusion, it can be stated that, after some stabilization time t_s, the stresses $\mathbf{S}$ and the plastic strain rates $\dot{\mathbf{E}}^p$ become periodic, with period Δt. Through a procedure similar to the previous one it can also be proven (but the proof is omitted for brevity) that—like for small displacements—the steady-state response is independent of any initial plastic strains (provided that these strains are within the D-stability limits of the structure).

The steady-state response, that is the response pertaining to the steady cycle, can in principle be determined by addressing an *ad hoc* equation set. To this purpose, using again the notation $t = \tau + t_{n-1}$, $t_n = n\Delta t$, $0 \leqslant \tau \leqslant \Delta t$, $n = 1, 2, \ldots$, let one remark that $\mathbf{S}(\mathbf{X}, \tau)$ and $\dot{\mathbf{E}}^p(\mathbf{X}, \tau)$ are independent of n [like the load $\mathbf{P}(\tau)$], whereas $\mathbf{E}(\mathbf{X}, \tau)$ and $\mathbf{u}(\mathbf{X}, \tau)$ depend both on τ and n. Then, the above equation set reads

$$\mathbf{F} = \mathbf{I} + \nabla\mathbf{u} \quad \text{in } V \tag{71a}$$

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{I}) \quad \text{in } V \tag{71b}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial_D V \tag{71c}$$

$$\operatorname{div}(\mathbf{S} \cdot \mathbf{F}^T) + \mathbf{b} = \mathbf{0} \quad \text{in } V \tag{72a}$$

$$\mathbf{n} \cdot \mathbf{S} \cdot \mathbf{F}^T = \mathbf{f} \quad \text{on } \partial_T V \tag{72b}$$

$$\dot{\mathbf{E}} = \mathbf{C}^{-1} : \dot{\mathbf{S}} + \dot{\mathbf{E}}^p \quad \text{in } V \tag{73}$$

$$\dot{\mathbf{E}}^p = \dot{\lambda} \frac{\partial\phi}{\partial\mathbf{S}} \quad \text{in } V \tag{74a}$$

$$\phi(\mathbf{S}) \leqslant 0, \quad \dot{\lambda} \leqslant 0, \quad \dot{\lambda}\phi(\mathbf{S}) = 0 \quad \text{in } V \tag{74b}$$

$$\Delta\mathbf{S} = \mathbf{0} \quad \text{in } V \tag{75}$$

all of which hold for $0 \leqslant \tau \leqslant \Delta t$ and for $n = 1, 2, 3, \ldots$

These equations must be solved sequentially for $n = 1, 2, \ldots$ to obtain the displacements and strain fields $\mathbf{u}(\mathbf{X}, t)$ and $\mathbf{E}(\mathbf{X}, t)$ for the whole cycle sequence, as well as the stress and plastic strain rate fields $\mathbf{S}(\mathbf{X}, \tau)$ and $\dot{\mathbf{E}}^p(\mathbf{X}, \tau)$. For greater clarity, this solution is referred to as the "limit solution" to the steady-state response problem. Through procedures like that used previously to demonstrate the existence of a steady cycle, it can be proven that the limit solution is unique for all, except the stress $\mathbf{S}$ in the elastic portion(s) (if any) of V, and also that it essentially coincides with the steady-state response. This statement is well known in the framework of small displacements [see, e.g. Polizzotto (1994)]; no formal proof of it is given here for brevity. As a consequence the properties of the steady cycle can be derived as the properties of the limit solution. In particular we can establish the following.

(1) The plastic strain ratchet, $\Delta\mathbf{E}^p(\mathbf{X})$, that is the plastic strain accumulated in every cycle, is compatible with the cycle displacement increments in the body's spatial configuration at the beginning of the cycle; that is, with reference to the nth cycle, $\Delta\mathbf{E}^p(\mathbf{X})$ is compatible with $\Delta\mathbf{u}_n(\mathbf{X}) := \mathbf{u}(\mathbf{X}, t_n) - \mathbf{u}(\mathbf{X}, t_{n-1})$ in the spatial configuration at time $t = t_{n-1}$. In fact, using the notation $(\cdot)_n := (\cdot)|_{t=n\Delta t}$ and $\Delta(\cdot)_n := (\cdot)_n - (\cdot)_{n-1}$, one can write

$$\Delta\mathbf{E} = \Delta\mathbf{E}^p \tag{76a}$$

$$= \tfrac{1}{2}(\mathbf{F}^T_{n-1} \cdot \Delta\mathbf{F}_n + \Delta\mathbf{F}^T_n \cdot \mathbf{F}_{n-1} + \Delta\mathbf{F}^T_n \cdot \Delta\mathbf{F}_n) \quad \text{in } V, n = 1, 2, \ldots \tag{76b}$$

$$\Delta\mathbf{u}_n = \mathbf{0} \quad \text{on } \partial_D V \quad n = 1, 2, \ldots \tag{76c}$$

Equation (76a) is derived from eqns (73) and (75), whereas eqns (76b,c) are derived from eqns (71b,c) written for $t = t_n$ and $t = t_{n-1}$ and subtracting them from each other.

(2) The fictitious volume and surface forces simulating the geometrical effects cumulated in every cycle identically vanish. In fact, one can write

$$\Delta \mathbf{b}_{Gn} := \mathrm{div}(\mathbf{S}_0 \cdot \Delta \mathbf{F}_n^T) = \mathbf{0} \quad \text{in } V, \quad n = 1, 2, \ldots$$

$$\Delta \mathbf{f}_{Gn} := -\mathbf{n} \cdot \mathbf{S}_0 \cdot \Delta \mathbf{F}_n^T = \mathbf{0} \quad \text{on } \partial_T V, \quad n = 1, 2, \ldots \tag{77}$$

Considering the body's steady-state response, the following is worth being noted. If $\dot{\mathbf{E}}^p = \mathbf{0}$ in V for all $t \geqslant t_s$, the steady cycle is elastic (elastic shakedown). On the other hand, if $\dot{\mathbf{E}}^p \neq \mathbf{0}$, but $\Delta \mathbf{E}^p = \mathbf{0}$ everywhere in V, $\mathbf{E}^p$ is periodic as periodic can be shown to be $\mathbf{u}$, and thus the body undergoes regular elastic–plastic oscillations around the "post-transient" spatial configuration $\mathbf{x}_0 = \mathbf{X} + \mathbf{u}_0(\mathbf{X})$ with initial and final stresses $\mathbf{S}_0$ in every cycle and $\Delta \mathbf{u}_n \equiv \mathbf{0}$, $\Delta \mathbf{F}_n \equiv \mathbf{0}$, in $V \cup \partial_T V$, $n = 1, 2, \ldots$ (alternating plasticity or plastic shakedown).

In the case that $\Delta \mathbf{E}^p$ is not identically vanishing, the steady cycle implies constant plastic strain growth cycle-by-cycle without stress changes (ratchetting, or incremental collapse). However, such a collapse mode can actually exist if, and only if, eqns (77) can be satisfied with $\Delta \mathbf{F}_n \neq \mathbf{0}$, i.e. $\Delta \mathbf{u}_n \neq \mathbf{0}$, in $V \cup \partial_T V$, $n = 1, 2, \ldots$, which in general can be excluded (except perhaps in particular structural conditions). This means that—differing from the small displacements case—ratchetting as a steady cycle cannot occur. This result, confirmed by several numerical analysis, deserves deeper study, which will be done elsewhere.

7. LACK OF STANDARD STABILITY

In the preceding developments, D-stability has been thought of to imply standard stability. Under this hypothesis, the elastic–plastic and fictitious elastic responses of a D-stable structure to a specified load history turn out to be uniquely determined, and this physical circumstance made it possible to simplify the phrasing in the presentation of the extended shakedown theorems. However, the above hypothesis is not necessary for the validity of these theorems. In fact, in the case of lack of standard stability, a fundamental equilibrium path of the structure may possess bifurcation points and hence stable or unstable equilibrium branches, and these branches must be included in the set of neighbor equilibrium paths in order to satisfy the D-stability criterion given in Section 3. Therefore, the fictitious elastic response considered with Melan's theorem of Section 3 must be intended not only as the fundamental equilibrium path associated with any ALH, but also any other related equilibrium branch, if any; the same holds in relation to the definition of PAM. With these extended validity limits, the results of Section 6 may constitute a theoretical confirmation of analogous results obtained by Dorosz (1993, private commun.) by means of pull–push strain-controlled experiments on elastic–plastic bars beyond their critical load limits.

Nevertheless, shakedown of a D-stable structure with lack of standard stability remains a problematic concept deserving adequate justification on physical grounds. For this reason, D-stability is intended as to encompass standard stability in this paper.

8. NUMERICAL APPLICATION

The concepts developed in the preceding sections for a continuous medium have been applied to a discrete structure as the plane truss shown in Fig. 1(a). In the Lagrangian approach to a truss structure (Cristfield, 1991), the only generalized forces to consider are the axial forces acting along the bar direction in the current deformed configuration. Since the bars remain rectilinear during deformation, equilibrium branches of the truss with one or more buckled bars are automatically ruled out, whereas those related to buckling of a global type are accounted for.

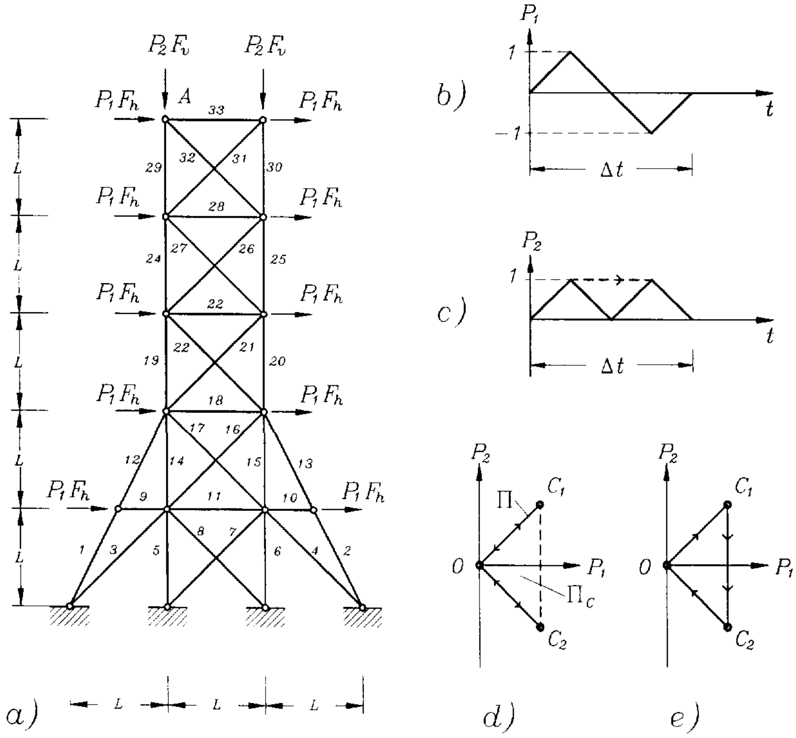

Fig. 1. Plane truss subjected to variable loads: (a) geometrical and loading scheme; (b) and (c) load history $P_1(t)$ and $P_2(t)$; (d) load domain Π in the (P_1, P_2)-plane; (e) load path enveloping the convex hull Π_c of Π.

The truss is subjected to horizontal (P_1F_h) loads and vertical (P_2F_v) loads as depicted in Fig. 1(a), where F_h and F_v are fixed, whereas P_1 and P_2 evolve periodically in time as shown in Figs 1(b,c); that is, in the (P_1, P_2)-plane, the point (P_1, P_2) describes the path OC_1OC_2O of Fig. 1(d) for every cycle. The bilateral segment C_1OC_2 of Fig. 1(d) constitutes the relevant load domain Π, whose convex hull Π_c is the triangle C_1OC_2. The material is elastic–perfectly-plastic with yield stress $S_y = 40$ kN cm^{-2}. The Young modulus is taken as low as $E_Y = 5000$ kN cm^{-2} in order to increase the system's deformability and consequent geometry change effects. The bar's cross-section areas are reported in Table 1.

The values $F_h = 50$ kN and $F_v = 750$ kN have been fixed for a first series of analyses. Primarily, by guess, the initial plastic strains $\bar{E}^p = -1.24\times10^{-2}$ for bar #25 and $\bar{E}^p = 0$ for all other bars were imposed. A (fictitious) elastic analysis under the above load history and initial plastic strains showed that the resulting bar stresses are below yield. By Melan Theorem, this result is sufficient to state that shakedown occurs. The shakedown occurrence has been also proved by a direct step-by-step elastic–plastic analysis for a load history of duration $4\Delta t$ (4 cycles), finding out that the bar plastic strains stabilize at constant values after some times t_s in the first cycle [as shown in Fig. 2(a) for bar #25, the only bar undergoing plastic strains], whereas the node displacements become periodic after t_s [as

Table 1. Bar's cross-section areas of the truss of Fig. 1(a)

Area (cm^2)	Bar numbers
10	27, 28, 29, 30, 31, 32, 33
20	1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 16, 17, 18, 23, 24, 25
30	21, 22
40	19, 20
50	5, 6, 14, 15

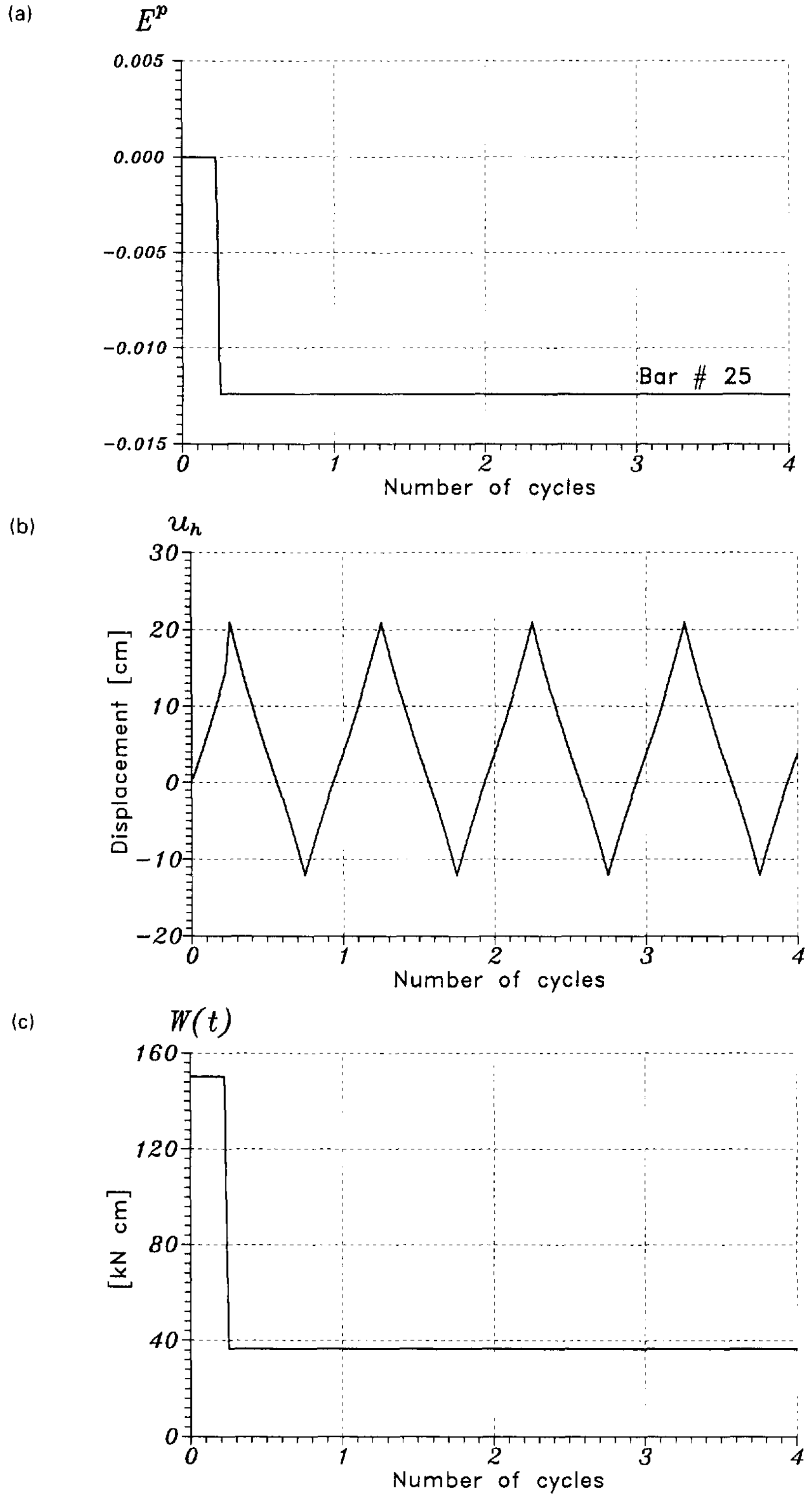

Fig. 2. Elastic–plastic response of the truss of Fig. 1(a) to a load history as in Figs 1(b,c) with $F_h = 50$ kN and $F_v = 750$ kN (i.e. below the elastic shakedown limit): (a) plastic strain history in the bar #25 (all the other bars remain elastic); (b) horizontal displacement history of node A of the truss; (c) D-stability functional W plotted as a function of time.

shown in particular for the left upper node A of the truss structure in Fig. 2(b)]. Furthermore, the D-stability functional $W(t)$ (computed considering the above elastic response to the loads and initial plastic strains $\bar{E}^p$ as a fundamental equilibrium path and the above elastic–plastic response as a neighbor equilibrium path) turns out to be positive for all t

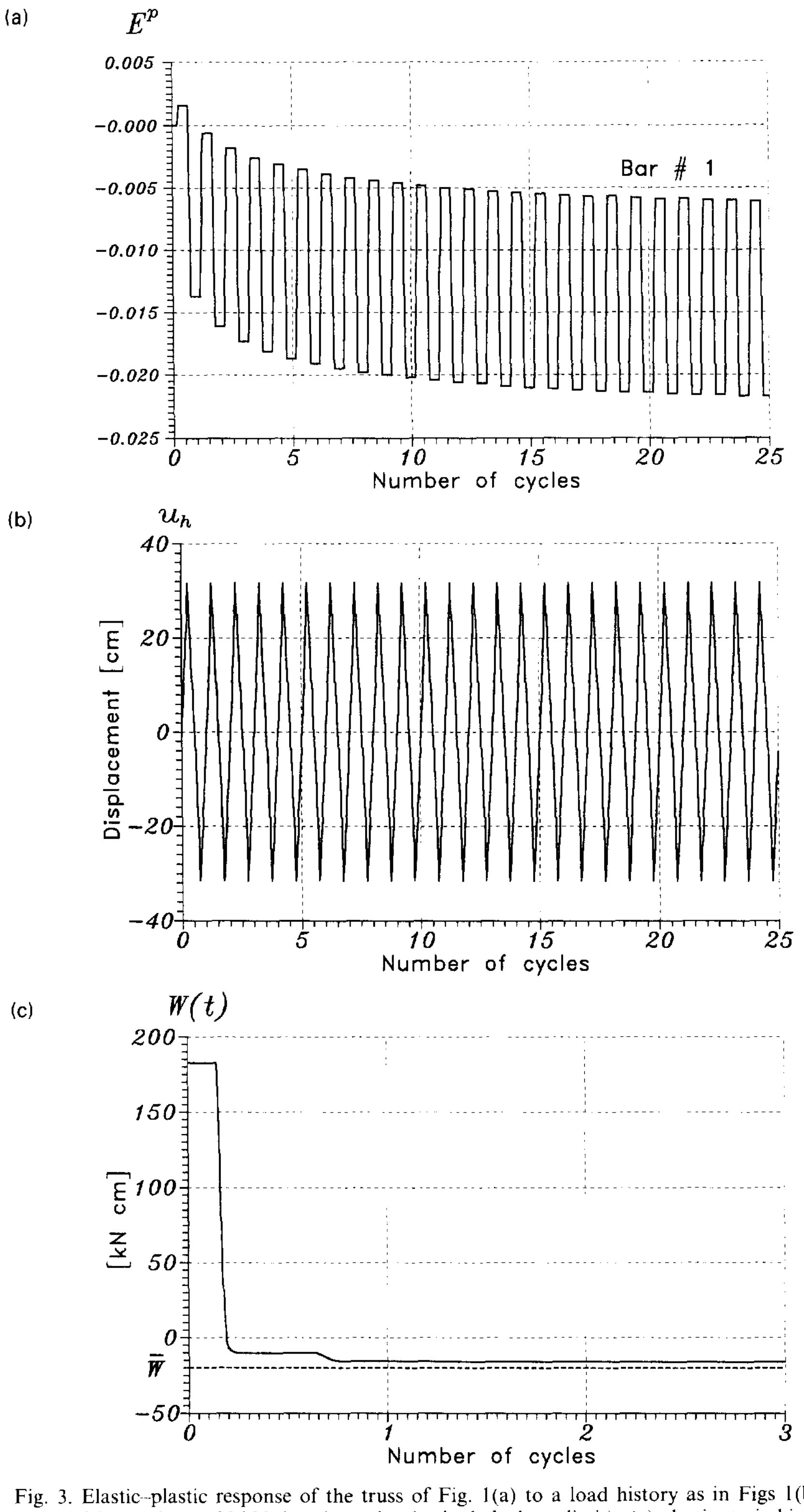

Fig. 3. Elastic–plastic response of the truss of Fig. 1(a) to a load history as in Figs 1(b,c) with $F_h = 100$ kN and $F_v = 20$ kN (i.e. above the elastic shakedown limit): (a) plastic strain histories for bar #1; (b) horizontal displacement history of node A of the truss; (c) D-stability functional W plotted as a function of time.

and constant for $t \geq t_s$, as shown in Fig. 2(c). An additional elastic–plastic analysis has been made considering the load path OC_1C_2O of Fig. 1(e), that is the load path enveloping the convex hull Π_c, with the result that elastic shakedown occurs also in this case.

A second series of analyses has been performed with $F_h = 100$ kN, and $F_v = 20$ kN.

This time the load history is above the elastic shakedown limit. The actual response has been found by a step-by-step elastic–plastic analysis for the first 25 cycles. Figure 3(a) reports the output plastic strain history for bars #1 (similar plots can be obtained for the bars #2, 12 and 13), whereas Fig. 3(b) reports the horizontal displacement history of node A. Additionally, the D-stability functional $W(t)$ has been computed taking the actual response (with zero initial plastic strain) as the fundamental equilibrium path and the same response shifted of Δt in the positive time axis as a neighbor equilibrium path. It is seen that the actual response tends almost asymptotically to a reverse plasticity behavior, while $W(t)$ becomes almost constant after a few cycles, with a lower bound $\bar{W} = -20$ kN cm.

9. DISCUSSION AND CONCLUSION

The large-displacement shakedown theory presented in the previous sections differs from the classical (small displacement) shakedown theory in some aspects which are worth note.

(1) Whereas the material stability postulate is sufficient to develop the classical shakedown theory, an additional stability principle at structural level, namely D-stability, must be used in the framework of large displacements.

(2) The (fictitious) elastic response of the body to the loads is crucial both for small- and large-displacement shakedown theories. However, whereas this response is not coupled with the analogous response to imposed plastic strains in the case of small displacements, this is not true for large displacements, with consequent conceptual and computational complications.

(3) Lower and upper bound theorems similar to those of classical shakedown theory are in principle available for the search of large-displacement shakedown loads, but their practical application is notably more complicated due to the geometric nonlinearity and consequent response coupling.

The proposed shakedown theory seems to possess the characteristics of a "sound" extension of classical shakedown theory to large displacements, and its greater complexity appears to be quite natural. Though the additive strain decomposition rule restricts the validity limits of the proposed shakedown theory to structural situations characterized by small strains and moderate rotations (pin-jointed structures, beam and frame structures, thin plate and shells), it is nevertheless believed that this theory provides an effective advance towards a satisfactory extended shakedown theory, as useful as the classic one for design purposes. The concept of D-stability has had a crucial role in the development of this extended shakedown theory. The assumption has been made that a D-stable structure, to which the extended shakedown theory is applicable, is also stable in the usual sense (no buckling); however, this perhaps is an unnecessary restriction that may be removed subsequently.

Differing from the small-displacement case, ratchetting as a steady-state response to periodic loads has been found to be impossible in the presence of geometry change effects (except perhaps in special situations). Such a phenomenon, confirmed by a number of numerical analyses, deserves deeper discussion.

Further study is also necessary in order to improve this theory and in particular to envisage numerical procedures for engineering applications. Additionally, the assessment of the D-stability limits for a given structure/load system, as well as the relationship between D-stability and stability in the usual sense, are points of study for future research work.

Note added in proof—In a paper (see ref. below) written before reading the proofs, it has been found that, for any structure stable in the large in the Drucker sense [i.e. $\Delta L_e > 0$ in eqn (20)], the D-stability functional $W(t)$ of eqn (19) is always bounded from below and thus, by Remark 4, such a structure is D-stable. This result implies that the extended shakedown theorems here presented apply, at least, to this class of structures. Polizzotto, C., Borino, G. and Fuschi, P. (1995). An extended shakedown theory for elastic–plastic-damage material models. *Eur. J. Mech. A/Solids* (in press).

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